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Tangent flag bundles and Jacobian varieties. Nota III

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Geometria. — *Tangent flag bundles and Jacobian varieties.* Nota III di SAMUEL A. ILORI e AUBREY W. INGLETON, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — Definiamo le sottovarietà «di Ehresmann» di un fascio di bandiere tangenti $V^{\triangle}$ sopra una varietà proiettiva algebrica irriducibile non-singolare, definita sopra un campo algebricamente chiuso. Poi mostriamo, usando una formula di intersezione, che le classi di cicli di tali sottovarietà «di Ehresmann» nell'anello di Chow di $V^{\triangle}$ possono essere determinate usando una conoscenza del più facile calcolo corrispondente in una varietà di bandiere F $(n+1)$. Questa teoria è poi applicata al calcolo delle classi di cicli di sottovarietà Jacobiane di V che sono definite mediante una famiglia indiciata di «nests» di sistemi lineari di «primals» in V.

4. THE INVARIANCE PRINCIPLE

In this section, we shall show that the cycle classes of 'Ehresmann' subvarieties of $V^{\triangle}$ in the Chow ring of $V^{\triangle}$ are determined by a knowledge of the easier corresponding Ehresmann classes of F $(n+1)$. This fact is one of the main goals of this work and we shall establish it by using a knowledge of 'Ehresmann' classes of codimension one found in Theorem 2.5 in Note I and the intersection formula proved in Theorem 3.3 in Note II.

THEOREM 4.1. (*The Invariance Principle*) (Cf. 2.5 *of* [6]). *For any given proper* (q, t)*-index* $\boldsymbol{k}$, *there is a polynomial* $J_{\boldsymbol{k}}(x_0, \cdots, x_q)$, (*depending only on* $\boldsymbol{k}$), *with integral coefficients such that, for any non-singular variety* V *of dimension* $\geq q$ *and any sufficiently general nest* $\mathscr{L}$ *of linear systems on* V *with top dimension* $\geq t-1$,

$$[\boldsymbol{k} ; \mathscr{L} \mid V^{\triangle}]^* = J_{\boldsymbol{k}}(\rho^* a, \delta_1, \cdots, \delta_q),$$

where $a \in A^1(V)$ *is the cycle class of the member of* $\mathscr{L}_1$, *and for any flag manifold* $F = F(n+1)$ *with* $n \geq t$,

$$[\boldsymbol{k} ; F]^* = J_{\boldsymbol{k}}(-\gamma_0, \gamma_0 - \gamma_1, \cdots, \gamma_0 - \gamma_q).$$

Proof. The proof depends on a series of inductions.

First we perform an induction on the codimension of the cycle class $[\boldsymbol{k} ; \mathscr{L} \mid V^{\triangle}]^*$. For codimension 1, the theorem follows from the fact that $[\boldsymbol{k} ; \mathscr{L} \mid V^{\triangle}]^*$ is one of

$$w(0 ; \mathscr{L} \mid V^{\triangle}), \cdots, w(q ; \mathscr{L} \mid V^{\triangle}),$$

(*) Nella seduta del 12 gennaio 1980.

and Theorem 2.5 gives the polynomials corresponding to the $w(i\,;\mathscr{L}\mid V^{\triangle})$ explicitly. We then assume that the theorem is proved for cycle classes of codimension less than the codimension of $[\boldsymbol{k}\,;\mathscr{L}\mid V^{\triangle}]^*$.

Next we perform an induction on q. When $q=0$, (i.e. when $\boldsymbol{k}=(k_0)$), we note that by the intersection formula (cf. Theorem 3.3),

$$[\boldsymbol{k}\,;\mathscr{L}\mid V^{\triangle}]^* = w(0\,;\mathscr{L}\mid V^{\triangle})\cdot[(k_0-1)\,;\mathscr{L}\mid V^{\triangle}]^*.$$

Hence by the first inductive hypothesis, the theorem is true for $[(k_0-1)\,;\mathscr{L}\mid V^{\triangle}]^*$ and this proves the theorem for $q=0$. We thus assume that the theorem is also true for all $[\boldsymbol{k}'\,;\mathscr{L}\mid V^{\triangle}]^*$ such that $\boldsymbol{k}'$ is a $(p\,,t)$-index $p\leq q-1$.

Now we are ready to prove the theorem for $\boldsymbol{k}=(k_0\,,\cdots,k_q)$. k_q cannot be zero, for if it were $\boldsymbol{k}$ will not be a proper index. Suppose $k_q=1$, then by the intersection formula,

$$w(q\,;\mathscr{L}\mid V^{\triangle})\cdot[(k_0\,,\cdots,k_{q-1}\,,0)\,;\mathscr{L}\mid V^{\triangle}]^* = [\boldsymbol{k}\,;\mathscr{L}\mid V^{\triangle}]^* +$$

$$+\ \text{other terms with indexes } (p\,,t)\,,\ p=q-1\,.$$

Hence by the second inductive hypothesis, the theorem is proved for $k_q=1$. Finally, we perform a third induction on the value of k_q. We assume that the theorem is proved for all $[\boldsymbol{k}'\,;\mathscr{L}\mid V^{\triangle}]^*$ such that $k'_q<k_q$. We, at once, observe that for k''_q as the biggest integer smaller than k_q not in $\{k_0\,,\cdots,k_q\}$,

$$w(q\,;\mathscr{L}\mid V^{\triangle})\cdot[(k_0\,,\cdots,k_{q-1}\,,k''_q)\,;\mathscr{L}\mid V^{\triangle}]^* = [\boldsymbol{k}\,;\mathscr{L}\mid V^{\triangle}]^* +$$

$$+\ \text{other terms with } k'_q=k''_q,\ \text{(by the intersection formula)}.$$

Hence by the third inductive hypothesis, the first part of the theorem is proved for all $\boldsymbol{k}$.

To prove the last part of the theorem, we let

$$\theta : V^{\triangle} \to W$$

be the natural injection associated with the inclusion $i : V \to \mathbf{P}_n$, (where $\alpha : W \to \mathbf{P}_n$ is the flag bundle, fibre $F(d+1)$ associated with the tangent bundle of $\mathbf{P}_n$). Then for the nest $\mathscr{L}'$, where $\mathscr{L}'_i$ is cut on V by the primes through E_{n-i} of the fixed flag

$$E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset \mathbf{P}_n$$

we have

$$\theta^*([\boldsymbol{k}\,;W]^*) = [\boldsymbol{k}\,;\mathscr{L}\mid V^{\triangle}]^*.$$

Since A(W) can be considered as a subring of A(F$(n+1)$), we have $[\boldsymbol{k}\,;W]^* = [\boldsymbol{k}\,;F]^*$ where $F=F(n+1)$. The result then follows from the fact that from the proof of Theorem 2.5,

$$\theta^* (\gamma_0 - \gamma_j) = \delta_j , \qquad j = 1 , \cdots , q$$

and

$$\theta^* (-\gamma_0) = \rho^* a ,$$

where a is the cycle class of the member of $\mathscr{L}'_1$.

5. Jacobian varieties

We shall conclude by giving an application of the invariance principle of Theorem 4.1 to show how to determine the cycle classes of Jacobian subvarieties of a non-singular irreducible algebraic projective variety V. Such Jacobians are projections on V of intersections of 'Ehresmann' subvarieties of $V^{\triangle}$, and are defined in terms of indexed families of nests of linear systems of primals on V. First we shall define Jacobian varieties which are generalizations of the classical Jacobian in its most general from (cfr [11], p. 22, [9]) as well as the 'generalized' Jacobian of linear systems of [7].

Definition 5.1. An *indexed family* of nests of linear systems comprises nests

$$\mathscr{L}^{(\alpha)} \colon \mathscr{L}_1^{(\alpha)} \subset \cdots \subset \mathscr{L}_{t_\alpha}^{(\alpha)} , \qquad (\alpha = 1 , \cdots , u)$$

together with, for each α, an (h_α , t_α)-index $\boldsymbol{k}^{(\alpha)}$, $0 \leq h_\alpha \leq d$. (See Definition 2.1 for the definition of an index). Then we define the *Jacobian* of the indexed family, which we assume to be sufficiently general, as the locus

$$\mathrm{J} = \mathrm{J} (\boldsymbol{k}^{(1)}, \cdots , \boldsymbol{k}^{(u)} ; \mathscr{L}^{(1)}, \cdots , \mathscr{L}^{(u)})$$

of points on V which are origins S_0 of tangent flags S to V satisfying all the conditions

$$\dim \mathscr{L}_i^{(\alpha)} (j , \mathrm{S}) \geq d_{ij} (\boldsymbol{k}^{(\alpha)}) - 1$$

$$((i , j) \in \mathrm{C} (\boldsymbol{k}^{(\alpha)}) , \alpha = 1 , \cdots , u) .$$

The determination of the Jacobian J is given by the following

Theorem 5.2. *The cycle class of a Jacobian, given in Definition 5.1, in the Chow ring* A (V) *of a non-singular algebraic irreducible projective variety* V *is determined by the Chow ring* A (F $(n+1)$) *of a flag manifold* F $(n+1)$ *for a sufficiently large* n *and by the Gysin homomorphism*

$$\rho_* : \mathrm{A} (\mathrm{V}^{\triangle}) \to \mathrm{A} (\mathrm{V})$$

for the tangent flag bundle $\rho : \mathrm{V}^{\triangle} \to \mathrm{V}$.

Proof. The Jacobian J is the projection on V of the subvariety

$$J^{\Delta} = \bigcap_{\alpha=1}^{u} [\boldsymbol{k}^{(\alpha)} ; \mathscr{L}^{(\alpha)} \mid V^{\Delta}]$$

of V^{Δ}. The indexed family is assumed to be sufficiently general for each component of J^{Δ} to have the correct dimension and to occur with multiplicity one in the intersection of the $[\boldsymbol{k}^{(\alpha)} ; \mathscr{L}^{(\alpha)} \mid V^{\Delta}]$. The cycle class j^{Δ} of J^{Δ} is then given by

$$j^{\Delta} = \prod_{\alpha=1}^{u} [\boldsymbol{k}^{(\alpha)} ; \mathscr{L}^{(\alpha)} \mid V^{\Delta}]^{*}.$$

By the invariance principle (cfr. Theorem 4.1), each cycle class $[\boldsymbol{k}^{(\alpha)} ; \mathscr{L}^{(\alpha)} \mid V^{\Delta}]^{*}$ can be obtained from $[\boldsymbol{k}^{(\alpha)} ; F]^{*}$ where $F = F(n+1)$ and $n \geq t$. Then the cycle class j of the Jacobian J is given by

$$j = \rho_{*}(j^{\Delta})$$

where ρ_{*} is the Gysin homomorphism $\rho_{*} : A(V^{\Delta}) \rightarrow A(V)$ for the tangent flag bundle $\rho : V^{\Delta} \rightarrow V$. The evaluation of the Gysin homomorphism ρ_{*} is stated as Theorem 2.2.1 in [6] for the complex case and is proved in [5] for the general case.

Remark. A special case of Definition 5.1 is as follows. Suppose we are given a (sufficiently general) family of linear systems (*not* nests) $\mathscr{L}^{(\alpha)}$, $\dim \mathscr{L}^{(\alpha)} = r_{\alpha}$, and integers p_{α}, $0 < p_{\alpha} < d$, $(\alpha = 1, \cdots, u)$. We define the $(p_1, \cdots, p_u)$-Jacobian of $\mathscr{L}^{(1)}, \cdots, \mathscr{L}^{(u)}$ to be the locus of origins S_0 of tangent flags S satisfying the conditions

$$\dim \mathscr{L}^{(\alpha)}(p_{\alpha}, S) \geq r_{\alpha} - 1, \qquad (\alpha = 1, \cdots, u).$$

Then the $(p, \cdots, p)$-Jacobian with $\Sigma r_{\alpha} = d - p + 1$ is classical; the $(1, \cdots, 1)$-Jacobian with no restriction on Σr_{α} was considered in [4].

In this special case,

$$\boldsymbol{k}^{(\alpha)} = (0, r_{\alpha} + 1, r_{\alpha} + 2, \cdots, r_{\alpha} + p_{\alpha}).$$

Hence from Lemma 2.4 and p. 358 of [12], we have

$$[\boldsymbol{k}^{(\alpha)} ; F]^{*} = (-1)^{p_{\alpha} r_{\alpha}} \begin{vmatrix} \sigma_{p_{\alpha}} & \cdots & \sigma_{p_{\alpha}-r_{\alpha}+1} \\ & \cdots & \\ \sigma_{p_{\alpha}+r_{\alpha}-1} & \cdots & \sigma_{p_{\alpha}} \end{vmatrix}$$

where σ_h denotes $\sigma_h(\gamma_0, \cdots, \gamma_{p_{\alpha}})$. That is

$$[\boldsymbol{k}^{(\alpha)} ; F]^{*} = (-1)^{p_{\alpha} r_{\alpha}} (\gamma_0 \gamma_1, \cdots, \gamma_{p_{\alpha}})^{r_{\alpha}} \bar{\sigma}_{r_{\alpha}} \left(\frac{1}{\gamma_0}, \cdots, \frac{1}{\gamma_{p_{\alpha}}} \right).$$

where $\bar{\sigma}_h(\gamma_0, \cdots, \gamma_q)$ is the h-th complete symmetric function in $\gamma_0, \cdots, \gamma_q$. By Theorem 4.1, $[\boldsymbol{k}^{(\alpha)}; \mathscr{L}^{(\alpha)} \mid V^{\triangle}]^*$ is obtained by substituting

$$-\gamma_0 = \rho^* a_\alpha \quad , \quad -\gamma_i = \rho^* a_\alpha + \delta_i , \qquad i = 1, \cdots, p_\alpha$$

in $[\boldsymbol{k}^{(\alpha)}; F]^*$, where $a_\alpha \in A^1(V)$ is the cycle class of a general member of $\mathscr{L}^{(\alpha)}$.

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