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# Section coefficients and section sequences

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**Algebra.** — *Section coefficients and section sequences.* Nota di
MARILENA BARNABEI, ANDREA BRINI e GIAN-CARLO ROTA (*), presentata (**) dal Socio G. ZAPPA.

RIASSUNTO. — Si studia una generalizzazione della nozione di coefficiente multinomiale, suggerita in parte da esempi combinatori studiati in precedenza, in parte dall'analogia con la teoria delle coalgebre. Si definisce la nozione di successione di tipo sezionale, che generalizza il concetto di carattere di un gruppo, e si dimostrano teoremi di finitezza relativi ad una algebra di operatori invarianti per traslazione ad essa associata.

## 1. INTRODUCTION

A fundamental problem of combinatorics is that of studying the ways of piecing together objects of given shapes to give an object whose shape is also preassigned, or else the inverse problem of finding ways and their number of splitting a given object into objects of given types. We introduce here two algebraic structures which we believe to be suggested by this problem, namely, *section coefficients* and *section sequences*. In this note we give, together with the definitions, some basic finiteness results and a few relevant examples.

We were led to these notions by previous work in two different subjects: the theory of coalgebras ([13]), further developed in the present spirit in [1] and [5], and the theory of polynomial sequences of binomial type ([8], [11], [12]).

## 2. BASIC DEFINITIONS

Given a set P whose elements will be called *pieces*, a *system of section coefficients* is, for each $n$, a function indicated by the symbol

$$(*) \qquad \begin{bmatrix} & a_0 & \\ a_1 & a_2 \cdots & a_n \end{bmatrix}$$

with $a_i \in \mathrm{P}$, taking values in a given field of characteristic zero (in fact, in most combinatorial examples taking non-negative integer values), to be read " the number of ways of splitting the piece of type $a_0$ into a sequence of pieces of types $a_1, a_2, \cdots, a_n$". These coefficients are subject to the following identities:

$$cs1) \quad \text{(consistency)}: \begin{bmatrix} a \\ b \end{bmatrix} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}, \qquad a, b \in \mathrm{P};$$

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*cs*2) (finiteness): for every $a_0 \in \mathrm{P}$ and for every $n \geq 1$ there is a finite number of sequences $a_1, a_2, \cdots, a_n$ such that (*) is not equal to zero;

*cs*3) (coassociative law): for every $n \geq 1$ we have:

$$\begin{bmatrix} & a_0 & \\ a_1\, a_2 & \cdots & a_n \end{bmatrix} = \sum_{p \in \mathrm{P}} \begin{bmatrix} & a_0 & \\ a_1\, a_2 & \cdots & a_k\, p \end{bmatrix} \begin{bmatrix} & p & \\ a_{k+1} & \cdots & a_n \end{bmatrix} =$$

$$= \sum_{p \in \mathrm{P}} \begin{bmatrix} & p & \\ a_1\, a_2 & \cdots & a_k \end{bmatrix} \begin{bmatrix} & a_0 & \\ p a_{k+1} & \cdots & a_n \end{bmatrix}, \qquad \text{for every} \quad k \leq n.$$

A related but less important concept is that of *cosection coefficients* on a set Q whose elements will be called *copieces*, in symbols:

$$(**) \qquad \begin{bmatrix} a_1\, a_2 & \cdots & a_n \\ & a_0 & \end{bmatrix}.$$

These satisfy the analogs of properties *cs*1), *cs*3) but, instead of *cs*2), they satisfy the following *cofiniteness property*: for every sequence $a_1, a_2, \cdots, a_n$ there is a finite number of $a_0$ such that (**) differs from zero.

Cosection coefficients (but not section coefficients) are nothing but the well-known structure coefficients of a ring.

The *empty piece* $\theta$, if any, is defined by the identity:

$$\begin{bmatrix} & a_0 & \\ a_1 \cdots a_k & \theta a_{k+1} & \cdots a_n \end{bmatrix} = \begin{bmatrix} & a_0 & \\ a_1 \cdots a_k & a_{k+1} & \cdots a_n \end{bmatrix},$$

and similarly the *empty copiece* (or zero) 0 is defined by

$$\begin{bmatrix} a_1 \cdots a_k & 0\, a_{k+1} & \cdots a_n \\ & a_0 & \end{bmatrix} = \begin{bmatrix} a_1 \cdots a_k & a_{k+1} & \cdots a_n \\ & a_0 & \end{bmatrix}.$$

A system of section coefficients is said to be *augmented* if there is a function $\epsilon$ (augmentation) defined on the pieces with scalar values such that:

$$\sum_{p \in \mathrm{P}} \begin{bmatrix} & a_0 & \\ a_1 \cdots a_k & p a_{k+1} & \cdots a_n \end{bmatrix} \epsilon(p) = \begin{bmatrix} & a_0 & \\ a_1 \cdots a_k & a_{k+1} & \cdots a_n \end{bmatrix}.$$

Finally, a system of section coefficients is said to be *well-augmented* when there is an augmentation $\epsilon$, an empty piece $\theta$, and $\epsilon(p) = 0$ if $p \neq \theta$, while $\epsilon(\theta) = 1$.

It is possible to associate a coalgebra with a distinguished basis to every augmented system of section coefficients (though we shall not do so here). By this association some of the results below can be used to yield simple proofs of several results on coalgebras.

Two guiding examples of section coefficients are the following:

a) *Boolean coefficients*: here, P is the family of all finite subsets of a given set, and

$$\begin{bmatrix} A_0 \\ A_1 A_2 \cdots A_n \end{bmatrix} = \begin{cases} 1 & \text{if } A_1 \cup A_2 \cup \cdots \cup A_n = A_0 \quad \text{and} \quad A_i \cap A_j = \varnothing \text{ for } i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

An analogous definition without finiteness conditions gives an example of cosection coefficients, which we call *coboolean coefficients*. In both cases both the empty piece and the empty copiece are the empty set.

b) *Partition coefficients*: let S be an infinite set. Consider the family $\Pi$ (S) of all partitions $\pi, \sigma, \tau, \cdots$ of finite subsets of S, ordered by refinement (thus, two comparable partitions have the same support). The set P consists of all ordered pairs $[\pi, \sigma]$ in $\Pi$(S) with $\pi \leq \sigma$ (that is, $\pi$ is finer that $\sigma$). Set

$$\begin{bmatrix} [\pi, \sigma] \\ [\pi', \tau]\ [\tau', \sigma'] \end{bmatrix} = \begin{cases} 1 & \text{if } \pi = \pi' \ , \quad \sigma = \sigma' \ , \quad \tau = \tau' \\ 0 & \text{otherwise.} \end{cases}$$

These examples are the simplest of a host of combinatorial examples generalizing the notion of hereditary bialgebras introduced in [7].

Actually, in most examples several pieces can be used interchangeably. This leads us to introduce an appropriate equivalence relation on a set of pieces. Say that an equivalence relation $\sim$ on the set of pieces P is *admissible* whenever

$$\sum_{b_i \in \bar{b}_i} \begin{bmatrix} a_1 \\ b_1\, b_2 \cdots b_n \end{bmatrix} = \sum_{b_i \in \bar{b}_i} \begin{bmatrix} a_2 \\ b_1\, b_2 \cdots b_n \end{bmatrix} \qquad \text{with} \quad a_1 \sim a_2 ,$$

where we indicate by $\bar{p}$ the equivalence class of the piece $p$.

Under these conditions one defines a system of section coefficients on the quotient set $P_{/\sim} = \overline{P}$, called the *reduced system*, by setting:

$$\begin{bmatrix} \bar{a} \\ \bar{b}_1\, \bar{b}_2 \cdots \bar{b}_n \end{bmatrix} = \sum_{b_i \in \bar{b}_i} \begin{bmatrix} a \\ b_1\, b_2 \cdots b_n \end{bmatrix}, \qquad a \in \bar{a} .$$

We give three examples of reduced systems: the first is *complete reduction*. Here, one considers all bijective functions $f: P \to P$ such that

$$\begin{bmatrix} a \\ b_1\, b_2 \cdots b_n \end{bmatrix} = \begin{bmatrix} f(a) \\ f(b_1)\, f(b_2) \cdots f(b_n) \end{bmatrix}$$

and one sets $p \sim q$ whenever $f(p) = q$ for at least one such functions.

The second is the special case of complete reduction for the Boolean system. The reduced section coefficients turn out to be the ordinary binomial and multinomial coefficients.

The third is obtained from partitions, setting two segments $[\pi, \sigma]$ and $[\pi', \sigma']$ to be equivalent whenever they are isomorphic as partially ordered sets. It turns out that each equivalence class (or *type*) is characterized by a vector $\lambda = (n_2, n_3, \cdots)$, where $n_i$ is the number of blocks of $\sigma$ which are cut into $i$ blocks by $\pi$. Under this admissible equivalence relation the reduced section coefficients are the *Faà di Bruno coefficients*:

$$\begin{bmatrix} & \lambda & \\ \mu & & \nu \end{bmatrix} = \begin{array}{l} \text{number of refinements } \pi \text{ of type } \mu \text{ of a partition } \sigma \\ \text{of type } \lambda, \text{ such that } [\pi, \sigma] \text{ is of type } \nu. \end{array}$$

In the particular case where $\lambda$ is the type of a partition having only one block with $n$ elements, denoted by $\delta_n$, one computes:

$$\begin{bmatrix} & \delta_n & \\ \mu & & \nu \end{bmatrix} = \frac{n!}{n_2!\, n_3! \cdots (2!)^{n_2} (3!)^{n_3} \cdots (n - 2\, n_2 - 3\, n_3 - \cdots)!},$$

where $\mu = (n_2, n_3, \cdots)$, provided that $\nu$ is the type of a partition having only one block, of cardinality $n_2 + n_3 + \cdots$, and zero otherwise.

## 3. Finiteness theorems

For a fixed piece $i$ and fixed integers $n$ and $r$, with $r < n$, we denote by $L_{n,r}(i)$ the set of all functions $\alpha(j_1, j_2, \cdots, j_r)$ with values in K of $r$ variables $j_1, j_2, \cdots, j_r$ ranging over P, satisfying the following linear equations:

$$\sum_{j_1, \cdots, j_r} \alpha(j_1, j_2, \cdots, j_r) \begin{bmatrix} i \\ b_1 \cdots b_{k_1}\, j_1\, b_{k_1+1} \cdots b_{k_2}\, j_2\, b_{k_2+1} \cdots b_{k_r}\, j_r\, b_{k_r+1} \cdots b_{k_{r+1}} \end{bmatrix} = 0$$

for every $b_1, \cdots, b_{k_1}, b_{k_1+1}, \cdots, b_{k_{r+1}}$ in P and for every choice of places $k_1, k_2, \cdots, k_{r+1}$ (note that $k_{r+1} + r = n$). We call these the *structural equations* of the system of section coefficients.

We also set $S_n(i)$ to be the set of all pieces $p$ such that

$$\begin{bmatrix} i \\ b_1 \cdots b_k\, p\, b_{k+1} \cdots n_n \end{bmatrix} \neq 0$$

for some choice of $b_1, \cdots, b_k, b_{k+1}, \cdots, b_n$ in P and some place $k$. Finally, we set $S(i) = \bigcup_{n \geq 2} S_n(i)$, and call $S(i)$ the *support* of the piece $i$.

Theorem 1.

*a*) for $n \geq 2r + 1$ we have

$$L_{n,r}(i) \subseteq L_{n+1,r}(i);$$

*b*) similarly we have

$$S(i) = S_2(i) \cup S_3(i),$$

in other words, every piece $i$ has finite support $S(i)$;

*c*) if the system is augmented, then

$$\mathrm{S}(i)=\mathrm{S}_3(i)\,;$$

*d*) if the system is augmented, then

$$\mathrm{L}_{n+1,r}(i)\subseteq \mathrm{L}_{n,r}(i) \qquad \text{for} \quad n\geq r\,;$$

*e*) for augmented systems and for $n\geq 2r+1$ we have:

$$\mathrm{L}_{n,r}(i)=\mathrm{L}_{n+1,r}(i)\,;$$

*f*) if the section coefficients have non-negative real values, then, setting $i\leq j$ if either $i=j$ or $i\in \mathrm{S}(j)$, then the relation $\leq$ is a quasi-order.

## 4. SECTIONS SEQUENCES

We now take a set P of pieces and a set Q of copieces; a sequence of scalar-valued functions $p_i(x)$ with $x\in \mathrm{Q}$, indexed by the pieces $i$, is said to be a *section sequence* when the functions $p_i$ are linearly independent and they satisfy the identity

$$\sum_{j,k\in \mathrm{P}}\begin{bmatrix} & i & \\ j & & k\end{bmatrix} p_j(x)\,p_k(y)=\sum_{z\in \mathrm{Q}}\begin{bmatrix} x & & y\\ & z & \end{bmatrix} p_i(z)$$

for every $i\in \mathrm{P}$ and $x\,,y\in \mathrm{Q}$.

This concept originates from the following classical special cases: Let Q be a group and set

$$\begin{bmatrix} x & & y\\ & z & \end{bmatrix}\neq 0\,, \qquad x\,,y\,,z\in \mathrm{Q}$$

whenever $xy=z$. Choose P to be the set of characters of the group, and set

$$\begin{bmatrix} & i & \\ j & & k\end{bmatrix}=1$$

if and only if $i=j=k$; then, a section sequence is simply a character of the monoid Q. Thus, section sequences generalize the notion of group character. A particular case that has been widely studied is that of polynomial sequences of binomial type, namely, real-valued polynomials $p_i(x)$ of degree $i$ with $x$ real, satisfying the identity:

$$p_i(x+y)=\sum_{j+k=i}\binom{i}{j}\,p_j(x)\,p_k(y)\,.$$

These are section sequences, the copieces being the real numbers with

$$\begin{bmatrix} x & & y\\ & z & \end{bmatrix}=1$$

if and only if $z = x + y$, and the pieces being the integers with the binomial coefficients as above.

By way of a mere example we define a section sequence analogous to the preceding for the Faà di Bruno system. Let Q be the monoid of formal power series:

$$f(t) = \sum_{i \geq 1} \frac{x_i}{i!} t^i \quad , \quad x_1 = 1$$

under functional composition, and set

$$\begin{bmatrix} f & & g \\ & h & \end{bmatrix} = \begin{cases} 1 & \text{if} \quad g \cdot f = h \\ 0 & \text{otherwise.} \end{cases}$$

With P as above, we say that a sequence $p_\lambda(f)$ is a *Faà di Bruno sequence* when it satisfies

$$\sum_{\mu,\nu} \begin{bmatrix} & \lambda & \\ \mu & & \nu \end{bmatrix} p_\mu(f)\, p_\nu(g) = p_\lambda(g \cdot f).$$

Thus, a Faà di Bruno sequence is a section sequence with formal power series as copieces and vectors $\lambda$ as pieces. The simplest Faà di Bruno sequence is obtained by setting

$$p_\lambda(f) = x_2^{n_2} x_3^{n_3} \cdots$$

if

$$f = \sum_{i \geq 1} \frac{x_i}{i!} t^i \qquad \text{and} \quad \lambda = (n_2, n_3, \cdots)$$

where $x_i$ are algebraic trascendentals for $i \geq 2$. From this example the purpose of the present definition should become clearer.

We intend to develop for Faà di Bruno sequences, as well as for more general section sequences, a functional calculus similar to the umbral calculus for sequences of polynomials of binomial type. This note is meant as a first step in this direction.

## 5. Operator calculus

We define the *left* and *right shift operators* $F_x$ and $E_x$ relative to a given section sequence as follows:

$$F_x p_i(y) = \sum_{z \in Q} \begin{bmatrix} y & & x \\ & z & \end{bmatrix} p_i(z)$$

$$E_x p_i(y) = \sum_{z \in Q} \begin{bmatrix} x & & y \\ & z & \end{bmatrix} p_i(z)$$

for every $i$ in P and for every $y$ in Q.

Denoting by P the vector space spanned by the section sequence $p_i(x)$, $i \in P$, we say that an operator $T : P \to P$ is *left* (or *right*) *shift-invariant* when T commutes with all the $F_x$ or all the $E_x$, respectively. Shift-invariant operators form an algebra. Assuming that the system is augmented, one can show that each of these algebras is isomorphic to the umbral algebra. This is the algebra of linear functionals $L, M, \cdots$ on P with multiplication $*$ (*convolution*) defined as

$$\langle L * M \mid p_i \rangle = \sum_{j,k \in P} \begin{bmatrix} & i & \\ j & & k \end{bmatrix} \langle L \mid p_j \rangle \langle M \mid p_k \rangle .$$

One can show that convolution is independent of the choice of the section sequence.

Among shift-invariant operators we note (in analogy with the boson calculus) the *left* and *right annihilation operators* $A_k$ and $B_k$, defined as follows:

$$A_k p_i(x) = \sum_{j \in P} \begin{bmatrix} & i & \\ k & & j \end{bmatrix} p_j(x)$$

$$B_k p_i(x) = \sum_{j \in P} \begin{bmatrix} & i & \\ j & & k \end{bmatrix} p_j(x)$$

for every $i$ in P and for every $x$ in Q.

Annihilation operators take the role played by derivatives in the special case of the umbral calculus, and the analog of Leibniz's rule for successive derivatives is

$$A_h A_k = \sum_{q \in P} \begin{bmatrix} & q & \\ k & & h \end{bmatrix} A_q .$$

Similarly, one can obtain an analog of Taylor's formula expressing every (left, say) shift-invariant operator T as a formal power series in the annihilation operators:

$$T = \sum_{i \in P} a_i B_i \quad , \quad a_i = T p_i(0) ,$$

assuming the existence of zero.

In particular, for the shift operators $F_y$ one obtains

$$F_y p_i(x) = \sum_{j \in P} p_j(y) B_j p_i(x)$$

in close analogy with Taylor's formula.

Sections sequences enjoy finiteness properties under translation reminiscent of polynomials and exponentials:

THEOREM 2.

*a*) Let $p_i(x)$, $i \in P$ be a section sequence; then for every $i \in p$ the subspace of P spanned by $E_y p_i(x)$, as $y$ ranges over Q, is finite-dimensional;

*b*) under the same assumptions, the subspace spanned by $A_j \, p_i(x)$, as $j$ ranges over P, is finite-dimensional.

For the Faà di Bruno system it turns out that right annihilation operators are products of the special annihilation operators $B_h$:

$$B_h \, p_\lambda(f) = \sum_\mu \begin{bmatrix} & \lambda & \\ \mu & & \delta_h \end{bmatrix} p_\mu(f) \, .$$

These turn out to be derivations: if $\lambda = (a_m, a_n)$, say, with $a_m = 1 = a_n$, and $f(t) = \sum_k \frac{x_k}{k!} t^k$, then

$$B_h \, p_\lambda(f) = B_h \, x_m \, x_n = x_n \, B_h \, x_m + x_m \, B_h \, x_n \, .$$

## References


[1] M. Barnabei, A. Brini e G.-C. Rota (1979) - *Sistemi di coefficienti sezionali*, « Centro di Analisi Globale », Firenze.

[2] P. Doubilet, G.-C. Rota and R. Stanley (1972) - *On the Foundations of Combinatorial Theory VI: the idea of generating function*, « Sixth Berkeley Symposium on Mathematical Statistics and Probability », *2*, Berkeley University Press, 267–318.

[3] F. Faà di Bruno (1876) - *Théorie des formes binaires*, Turin, Libr. Brero.

[4] A. M. Garsia (1973) - *An exposé of the Mullin-Rota theory of polynomials of binomial type*, « J, Linear and Multilinear Algebra », *1*, 43.

[5] M. Henle (1975) - *Binomial enumeration on dissects*, « Trans. Am. Math. Soc. », *202*, 1–38.

[6] S. A. Joni (1977) - *Polynomials of binomial type and the Lagrange inversion formula* « Ph. D. thesis », University of California.

[7] S. A. Joni and G.-C. Rota - *Coalgebras and bialgebras in combinatorics*, « Studies in Applied Mathematics », *61* (2), 1979.

[8] R. Mullin and G.-C. Rota (1970) - *On the Foundations of Combinatorial Theory III: theory of binomial enumeration, Graph Theory and its Applications*, « Academic Press », New York and London, 167–213.

[9] G. Polya (1956) - Picture writing, American Math. Monthly, *63*, 689.

[10] S. Roman (1979) - *The algebra of formal series*, « Advances in Mathematics », *31*, (3), 309–329.

[11] S. Roman and G.-C. Rota (1978) - *The Umbral Calculus*, « Advances in Mathematics », *27*, (2) 95–188.

[12] G.-C. Rota, D. Kahaner and A. Odlyzko (1973) - *Finite Operator Calculus*, « J. Math. Appl. », *42*, 685–760.

[13] M. E. Sweedler (1969) - *Hopf Algebras*, W. A. Benjamin, New York.

[14] W. T. Tutte (1967) - *On dichromatic Polynomials*, « J. Combinatorial Theory », *2*, 301.