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Tangent flag bundles and Jacobian varieties. Nota II

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Geometria. — *Tangent flag bundles and Jacobian varieties.* Nota II di SAMUEL A. ILORI e AUBREY W. INGLETON, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — Definiamo le sottovarietà «di Ehresmann» di un fascio di bandiere tangenti $V^{\triangle}$ sopra una varietà proiettiva algebrica irriducibile non - singolare, definita sopra un campo algebricamente chiuso. Poi mostriamo, usando una formula di intersezione, che le classi di cicli di tali sottovarietà «di Ehresmann» nell'anello di Chow di $V^{\triangle}$ può essere determinato usando una conoscenza del più facile calcolo corrispondente su una varietà di bandiere $F(n+1)$. Questa teoria è poi applicata al calcolo delle classi di cicli di sottovarietà Jacobiane di V che sono definite mediante una famiglia indiciata di «nests» di sistemi lineari di «primals» in V.

3. THE INTERSECTION FORMULA

In this section, we shall prove one of the main results of this paper. It is the intersection formula which gives the intersection of any 'Ehresmann' class $[\boldsymbol{k};\mathscr{L}\mid V^{\triangle}]^*$ with one of the classes $w(q;\mathscr{L}\mid V^{\triangle})$ of codimension one. Monk in [8] proved a similar intersection formula for the flag manifold $F(n+1)$. To prove the formula, we shall first find a specialisation of the linear systems involved in the definition of the 'Ehresmann' subvariety $[\boldsymbol{k};\mathscr{L}\mid V^{\triangle}]$. We shall then use the specialisation to break the intersection into components and conclude the proof by finding the multiplicities with which all the components occur. But first we need two lemmas about indices.

Let $\bar{C}(\boldsymbol{k})$ be the set of pairs (i,j) satisfying conditions (i), (iii), (iv) only of Definition 2.1. Also for such pairs $(i,j)\in\bar{C}(\boldsymbol{k})$, put

$$\bar{d}_{ij}(\boldsymbol{k}) = |\{0,\cdots,i\}\setminus\{k_0,\cdots,k_j\}|.$$

Let

$$C_1(\boldsymbol{k}) = \{i:(i,j)\in C(\boldsymbol{k})\} \quad , \quad C_2(\boldsymbol{k}) = \{j:(i,j)\in C(\boldsymbol{k})\}$$

$$\bar{C}_1(\boldsymbol{k}) = \{i:(i,j)\in\bar{C}(\boldsymbol{k})\} \quad , \quad \bar{C}_2(\boldsymbol{k}) = \{j:(i,j)\in\bar{C}(k)\}.$$

Note that $C_1(\boldsymbol{k})\subseteq\bar{C}_1(\boldsymbol{k})$ and $C_2(\boldsymbol{k})=\bar{C}_2(\boldsymbol{k})$. The following lemma shows that if $j\in C_2(\boldsymbol{k})$ or $\bar{C}_2(\boldsymbol{k})$, then any condition imposed on flags by a pair $(i,j)\in\bar{C}(\boldsymbol{k})$ is implied by a condition $(i',j)\in C(\boldsymbol{k})$.

(*) Nella seduta del 15 dicembre 1979.

LEMMA 3.1. *Let* $\boldsymbol{k} = (k_0, \cdots, k_q)$ *be an index and let* $(t, j) \in \bar{C}(\boldsymbol{k})$. *Then if* $(k_0, \cdots, k_j) \neq (0, \cdots, j)$

$$d_{sj}(\boldsymbol{k}) = \bar{d}_{tj}(\boldsymbol{k}),$$

where $s\ (s \leq t)$ *is the greatest member of* $\{k_0, \cdots, k_j\}$ *such that* $(s, j) \in C(\boldsymbol{k})$.

Proof. If $(t, j) \in C(\boldsymbol{k})$, then $s = t$ and the lemma follows. If $(t, j) \notin C(\boldsymbol{k})$, then it follows that $t - 1 \in \{k_0, \cdots, k_j\}$. Hence $(t - 1, j) \in \bar{C}(\boldsymbol{k})$ and

$$d_{t-1,j}(\boldsymbol{k}) = \bar{d}_{tj}(\boldsymbol{k}).$$

If $(t - 1, j) \in C(\boldsymbol{k})$, then the lemma also follows. If $(t - 1, j) \notin C(\boldsymbol{k})$, continue as above. After a finite number of steps, we have that $(i, j) \in \bar{C}(\boldsymbol{k})$ and $(i, j) \notin C(\boldsymbol{k})$ for $s < i \leq t$ (since $(k_0, \cdots, k_j) \neq (0, \cdots, j)$, but $(s, j) \in C(\boldsymbol{k})$. Since in this case

$$\{s, s + 1, \cdots, t\} \subseteq \{k_0, \cdots, k_j\}$$

it follows that

$$d_{sj}(\boldsymbol{k}) = \bar{d}_{tj}(\boldsymbol{k}).$$

This completes the proof of the lemma.

Remark. The above lemma shows that both $C(\boldsymbol{k})$ and $\bar{C}(\boldsymbol{k})$ give rise to the same 'Ehresmann' subvariety but that $C(\boldsymbol{k})$ gives the minimum independent conditions to be satisfied.

The following lemma, which is a technical one about indices, will be needed in the proof of the intersection formula.

LEMMA 3.2. *Let* $\boldsymbol{k} = (k_0, \cdots, k_q)$ *be an index such that*

$$C_1(\boldsymbol{k}) = \{i_0, \cdots, i_m\}$$

is in ascending order and let $\boldsymbol{k}' = (k'_0, \cdots, k'_q)$ *be another index got from* $\boldsymbol{k}$ *by replacing* $i_h + d_{h+1} - d_h - 1\ (\equiv s - 1)$ *in* $\boldsymbol{k}$, *where*

$$d_h = |\{0, \cdots, i_h - 1\} \cap \{k_0, \cdots, k_q\}|,$$

by the smallest integer $s' \geq s$ *which is not in* $\{k_0, \cdots, k_q\}$. *Then we have the following*

(i) $C_1(\boldsymbol{k}') \cup \{i_h\} \supseteq C_1(\boldsymbol{k}) \cup \{s\}$,

(ii) *If* $s = s'$ *and* $(s, j) \in C(\boldsymbol{k}')$, *then* $(i_h, j) \in C(\boldsymbol{k})$ *and*

$$d_{sj}(\boldsymbol{k}') = d_{i_h j}(\boldsymbol{k}) + 1.$$

(iii) *If* $s \neq s'$ *and for any* j, *such that* k'_j *comes before* s' *and* $(s, j) \in C(\boldsymbol{k}')$, *then* $(s, j) \in C(\boldsymbol{k})$ *and*

$$d_{sj}(\boldsymbol{k}') = d_{sj}(\boldsymbol{k}).$$

(iv) *If* $s \neq s'$ *and for any* j, *such that* $k'_j = s'$ *or* k'_j *comes after* s' *and* $(s, j) \in C(\boldsymbol{k}')$, *then* $(i_h, j) \in \overline{C}(\boldsymbol{k})$ *and*

$$d_{sj}(\boldsymbol{k}') = \bar{d}_{i_h j}(\boldsymbol{k}) + 1 .$$

Proof.

(i) Note that the number of elements in $\{k_0, \cdots, k_q\}$ which are not greater than i_h but less than i_{h+1} is $d_{h+1} - d_h - 1$ and they are precisely

$$\{i_h + 1, i_h + 2, \cdots, i_h + d_{h+1} - d_h - 1 \equiv s - 1\} .$$

Thus the following numbers occur in $\boldsymbol{k}'$ in the order shown

$$i_h, \cdots, i_h + 1, \cdots, i_h + d_{h+1} - d_h - 2, \cdots, s' .$$

Hence since $s - 1 \notin \{k'_0, \cdots, k'_q\}$ and $s \in \{k'_0, \cdots, k'_q\}$, then $s \in C_1(\boldsymbol{k}')$. Also if $s - 1 \neq i_h$, then $s - 1 \notin C_1(\boldsymbol{k})$ and $i_h \in C_1(\boldsymbol{k}')$. But if $s - 1 = i_h$, then $i_h \notin C_1(\boldsymbol{k}')$.

(ii) If $s = s'$ with $(s, j) \in C(\boldsymbol{k}')$, then

$$i_h \in \{k_0, \cdots, k_j\} \quad \text{and} \quad i_h - 1 \notin \{k_0, \cdots, k_j\} .$$

Also $s > i_h$ which implies that $k'_j \geq i_h$; and since either $k'_j = k_j$ or $k'_j = s$, then $k_j \geq i_h$. Now

$$\{i_h, i_h + 1, \cdots, i_h + d_{h+1} - d_h - 1\} \subseteq \{k_0, \cdots, k_j\} \quad \text{and} \quad k'_{j+1} = k_{j+1} < s$$

imply that $k_{j+1} < i_h$. Hence $(i_h, j) \in C(\boldsymbol{k})$. Now since

$$\{i_h, i_h + 1, \cdots, i_h + d_{h+1} - d_h - 2, s\} \subseteq \{k'_0, \cdots, k'_j\} ,$$

we have

$$d_{sj}(\boldsymbol{k}') = d_{j_h i}(\boldsymbol{k}) + 1 .$$

(iii) If $s \neq s'$ and for any j, such that k'_j comes before s' and $(s, j) \in C(\boldsymbol{k}')$, then

$$\{k'_0, \cdots, k'_j\} = \{k_0, \cdots, k_j\} .$$

k_{j+1} cannot be equal to s'. Hence $(s, j) \in C(\boldsymbol{k})$ and

$$d_{sj}(\boldsymbol{k}') = d_{sj}(\boldsymbol{k}) .$$

(iv) Now suppose $s \neq s'$ and $(s, j) \in C(\boldsymbol{k}')$ with $k'_j = s'$ or k'_j coming after s'. Then since i_h comes before s', we have

$$i_h \in \{k_0, \cdots, k_j\} .$$

Either $k_j = s - 1$ or $k_j = k'_j \geq s$ implies that $k_j \geq i_h$.

Now

$$k'_{j+1} = k_{j+1} \notin \{i_h, \cdots, s - 1, s\} \quad \text{and} \quad k'_{j+1} < s$$

imply that

$$k'_{j+1} = k_{j+1} < i_h .$$

Hence $(i_h , j) \in \overline{C}(\boldsymbol{k})$ and

$$d_{sj}(\boldsymbol{k}') = | \{0 , \cdots , i_h , i_h + 1 , \cdots , s\} \setminus \{k_0 , \cdots , k_{r-1} , s' , k_{r+1} , \cdots , k_j\} |$$
$$= | \{0 , \cdots , i_h\} \setminus \{k_0 , \cdots , k_j\} | + 1 , \qquad \text{since} \quad s - 1 \notin \{k'_0 , \cdots , k'_j\}$$
$$= \bar{d}_{i_h j}(\boldsymbol{k}) + 1 .$$

We now state the main result of this section as follows.

THEOREM 3.3. (*The Intersection Formula*) (Cfr. 2.3 *in* [6]). *For a sufficiently general nest of linear systems*

$$\mathscr{L} : \mathscr{L}_1 \subset \mathscr{L}_2 \subset \cdots \subset \mathscr{L}_t$$

we have, for any $(q , t - 1)$*-index* $\boldsymbol{k}$, $0 \leq q \leq d$,

$$w(q ; \mathscr{L} \mid V^{\triangle}) [\boldsymbol{k} ; \mathscr{L} \mid V^{\triangle}]^* = \Sigma \, [(k_0 , \cdots , k_{i-1} , k'_i , k_{i+1} , \cdots , k_q) ; \mathscr{L} \mid V^{\triangle}]^*$$

where for each $i = 0 , \cdots , q$, k'_i *is the smallest integer* $> k_i$ *which is not in* $\{k_0 , \cdots , k_q\}$ *and the summation is over all* i *such that there is no* h, $i < h < q$, *for which* $k_i < k_h < k'_i$.

Proof. The proof of the theorem will be in stages. First we shall find a specialisation for the linear systems involved in the definition of $[\boldsymbol{k} ; \mathscr{L} \mid V^{\triangle}]^*$ and $w(q ; \mathscr{L} \mid V^{\triangle})$. Note that $w(q ; \mathscr{L} \mid V^{\triangle})$ is the cycle class of the subvariety of flags satisfying the single condition

$$\dim \mathscr{L}_{q+1}(q , S) \geq 0 ,$$

where $\mathscr{L}_{q+1}$ is the linear system of primals on V of dimension q such that $\mathscr{L}_{q+1}$ is a subsystem of $\mathscr{L}_t$. Similarly, $[\boldsymbol{k} ; \mathscr{L} \mid V^{\triangle}]$ is a subvariety of flags satisfying the conditions

$$\dim \mathscr{L}_i(j , S) \geq d_{ij}(\boldsymbol{k}) - 1 , \qquad (i , j) \in C(\boldsymbol{k}) .$$

Let $C_1(\boldsymbol{k}) = \{i_0 , \cdots , i_m\}$ be in ascending order. For $h = 0 , \cdots , m$ let

$$d_h = | \{0 , \cdots , i_h - 1\} \cap \{k_0 , \cdots , k_q\} | .$$

We now specialise by putting

$$\mathscr{L}_{q+1} = \mathscr{L}'_{q+1} \quad , \quad \mathscr{L}'_{q+1} \cap \mathscr{L}_{i_h} = \mathscr{L}'_{d_h} \qquad (h = 1 , \cdots , m) ,$$

and if $d_0 > 0$,

$$\mathscr{L}'_{q+1} \cap \mathscr{L}_{i_0} = \mathscr{L}'_{d_0} .$$

Thus

$$\mathscr{L}_{i_{h-1}} + \mathscr{L}'_{d_h} = \mathscr{L}''_{i_{h-1}+d_h-d_{h-1}} \subseteq \mathscr{L}_{i_h}$$

and

$$\mathscr{L}'_{d_0} = \mathscr{L}''_{d_0} \subseteq \mathscr{L}_{i_0} .$$

We, therefore, have the following inclusions

$$\begin{array}{ccccccc} \mathscr{L}'_{d_0} \subset & \mathscr{L}'_{d_1} \subset & \cdots \subset & \mathscr{L}'_{d_m} \subset & \mathscr{L}'_{q+1} \\ \cap & \cap & & \cap & \cap \\ \mathscr{L}_{i_0} \subset & \mathscr{L}_{i_1} \subset & \cdots \subset & \mathscr{L}_{i_m} \subset & \mathscr{L}_t . \end{array}$$

The next stage of the proof is to use the above specialisation to break the intersection into components. A flag is thus in the intersection if S satisfies all the conditions defining $[\boldsymbol{k} ; \mathscr{L} \mid V^{\triangle}]$ and also

S_q is tangent to a member of $\mathscr{L}'_{q+1}$ but not of $\mathscr{L}'_{d_m}$,

or S_q is tangent to a member of $\mathscr{L}'_{d_m}$ but not of $\mathscr{L}'_{d_{m-1}}, \cdots$

or S_q is tangent to a member of $\mathscr{L}'_{d_1}$ but not of $\mathscr{L}'_{d_0}$,

or S_q is tangent to a member of $\mathscr{L}'_{d_0}$, if $d_0 > 0$.

The above conditions are mutually exclusive by the conditions of the specialisation.

For a general $h, -1 \leq h \leq m$, where we put $d_{m+1} = q + 1$ and $i_{-1} = 0 = d_{-1}$, consider the case where S_q is tangent to a member of $\mathscr{L}'_{d_{h+1}}$ but not of $\mathscr{L}'_{d_h}$. Thus S_j is tangent to a member of $\mathscr{L}'_{d_{h+1}}$ for all $j \leq q$ but not to any member of $\mathscr{L}'_{d_h}$. By the above specialisation,

$$\mathscr{L}_{i_h} + \mathscr{L}'_{d_{h+1}} = \mathscr{L}''_{i_h+d_{h+1}-d_h} \equiv \mathscr{L}''_s$$

and so a flag S is in this section of the intersection if S satisfies all the conditions satisfied by $[\boldsymbol{k} ; \mathscr{L} \mid V^{\triangle}]$ as well as

$$\dim \mathscr{L}''_s (j, S) \geq d_{i_h j} (\boldsymbol{k}), \qquad (i_h, j) \in \bar{C} (\boldsymbol{k}).$$

Now consider the index $\boldsymbol{k}'$ obtained from $\boldsymbol{k}$ by replacing $s - 1$ in $\boldsymbol{k}$ by the smallest integer $s' \geq s$ which is not in $\{k_0, \cdots, k_q\}$ and such that if $s - 1 = k_p$ there is no $r, p \leq r \leq q$, for which $s - 1 < k_r < s'$. This implies that if $s \neq s'$, $s' \notin C_1(\boldsymbol{k}')$ since then $s' - 1$ comes before s' in $\boldsymbol{k}'$. It follows from Lemma 3.2 that

$$\{i_h\} \cup C_1(\boldsymbol{k}') = C_1(\boldsymbol{k}) \cup \{s\}.$$

If $s - 1 \neq i_h$, then $C_1(\boldsymbol{k}') = C_1(\boldsymbol{k}) \cup \{s\}$. But if $s - 1 = i_h$, then $i_h \notin C_1(\boldsymbol{k}')$ and for any $(i_h, j) \in C(\boldsymbol{k})$,

$$\bar{d}_{s' j}(\boldsymbol{k}') \geq d_{i_h j}(\boldsymbol{k}).$$

Thus in this case any conditions imposed on flags by i_h in $[\boldsymbol{k} ; \mathscr{L} \mid V^{\triangle}]$ are implied by the conditions imposed on flags by s' in $[\boldsymbol{k}' ; \mathscr{L} \mid V^{\triangle}]$. From the above, one concludes that the component of the intersection is $[\boldsymbol{k}' ; \mathscr{L} \mid V^{\triangle}]$.

Finally, to conclude the proof, it remains to find the multiplicities with which the components occur in the intersection. Let U be a subvariety of V. Then we have the following diagram

$$\begin{array}{ccccc} j^{*}\,V^{\triangle} & \xrightarrow{\bar{\theta}} & V^{\triangle} & \xrightarrow{\theta} & W \\ \downarrow{\scriptstyle F(d)} & & \downarrow{\scriptstyle F(d)} & & \downarrow{\scriptstyle F(d+1)} \\ U & \xrightarrow{j} & V & \xrightarrow{i} & \mathbf{P}_n \end{array}$$

where $W \rightarrow \mathbf{P}_n$ is the flag bundle, fibre $F(d+1)$, associated with the tangent bundle $T(\mathbf{P}_n)$ of $\mathbf{P}_n$ and θ, $\bar{\theta}$ are injections. Thus $j^{*}V^{\triangle}$ is a subvariety of $V^{\triangle}$. When $\dim(U) = 1$ and if $e \in U$ is any point of U, then we have the diagram

$$\begin{array}{ccccccc} F = F(d) & \xrightarrow{\bar{\theta}_1} & j^{*}V^{\triangle} & \xrightarrow{\bar{\theta}} & V^{\triangle} & \xrightarrow{\theta} & W \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (e) & \longrightarrow & U_1 & \xrightarrow{j} & V & \xrightarrow{i} & \mathbf{P}_n \end{array}$$

$\bar{\theta}_1$ may be defined as

$$\bar{\theta}_1 (S_0 \subset \cdots \subset S_{d-1}) = (e) \subset (e) + S_0 \subset \cdots \subset (e) + S_{d-1}\,.$$

Note that

$$\bar{\theta}^{*}([\boldsymbol{k}\,;\mathscr{L} \mid V^{\triangle}]^{*}) = [\boldsymbol{k}\,;\mathscr{L}' \mid j^{*}V^{\triangle}]^{*}$$

where $\mathscr{L}'$ is the linear system cut on U_1 by the linear system on V, and

$$\bar{\theta}_1^{*}([\boldsymbol{k}\,;\mathscr{L} \mid j^{*}V^{\triangle}]^{*}) = \theta_1^{*}([\boldsymbol{k}\,;\mathscr{L}'' \mid j^{*}V^{\triangle}]^{*})\,,$$

for any linear systems $\mathscr{L}$, $\mathscr{L}''$ of primals on U_1. Hence

$$\begin{aligned} \bar{\theta}_1^{*}\,\bar{\theta}^{*}\,\theta^{*}([\boldsymbol{k}\,;W]^{*}) &= \bar{\theta}_1^{*}\,\bar{\theta}^{*}([\boldsymbol{k}\,;\mathscr{L}^{(')} \mid V^{\triangle}]^{*}) \\ &= \bar{\theta}_1^{*}([\boldsymbol{k}\,;\mathscr{L}^{('')} \mid V^{\triangle}]^{*}) \\ &= [\boldsymbol{k}'\,;F]^{*} \end{aligned}$$

where $\mathscr{L}^{(')}$, $\mathscr{L}^{('')}$ are linear systems cut on V and U_1 respectively by the primes through E_{n-i} of the fixed flag

$$E_0 \subset \cdots \subset E_{n-1} \subset \mathbf{P}_n$$

and where if $\boldsymbol{k} = (k_0, \cdots, k_h)$, then $\boldsymbol{k}' = (k_0 - 1, \cdots, k_h - 1)$. For instance

$$\bar{\theta}_1^{*}\,\bar{\theta}^{*}(w(q\,;\mathscr{L} \mid V^{\triangle})) = w(q-1\,;F)\,.$$

Thus if $\bar{\theta}^* ([\boldsymbol{k} ; \mathscr{L} \mid V^{\triangle}]^*) \neq 0$, then $\bar{\theta}_1^* \bar{\theta}^* (w (q ; \mathscr{L} \mid V^{\triangle}) . [\boldsymbol{k} ; \mathscr{L} \mid V^{\triangle}]^*)$ is the intersection on $F(d)$ of $w(q - 1 ; F)$ with an Ehresmann class in $A(F(d))$. Monk in [8] has proved that such an intersection splits into components with multiplicity 1 for the complex case but the proof is essentially the same for any algebraically closed field. $\bar{\theta}_1^*$ is the zero map only in the top dimension, i.e. in classes which are multiples of the class of a point in $A(j^* V^{\triangle})$. For dimension reasons, the intersection of such a class with $w(q ; \mathscr{L} \mid j^* V^{\triangle})$ is zero. Thus we have proved that if $\dim(U) = 1$, then the components of the intersection

$$\bar{\theta}^* (w (q ; \mathscr{L} \mid V^{\triangle}) \cdot [\boldsymbol{k} ; \mathscr{L} \mid V^{\triangle}]^*)$$

occur with multiplicity 1. We shall now prove unit multiplicity for $w(q ; \mathscr{L} \mid V^{\triangle}) \cdot [\boldsymbol{k} ; \mathscr{L} \mid V^{\triangle}]^*$.

Consider the diagram

$$\begin{array}{ccc} j^* V^{\triangle} & \xrightarrow{\bar{\theta}} & V^{\triangle} \\ \downarrow & & \downarrow \\ U_{d-1} & \xrightarrow{j} & V \end{array}$$

where $\dim(U) = d - 1$. Then by induction, one shows that the components of the intersection

$$\bar{\theta}^* (w (q ; \mathscr{L} \mid V^{\triangle}) \cdot [\boldsymbol{k} ; \mathscr{L} \mid V^{\triangle}]^*)$$

occur with multiplicity 1. In this case $\bar{\theta}^*$ is the zero map only in the top dimension, i.e. in classes of a point in $A(V^{\triangle})$. For dimension reasons, the intersection of such a class with $w(q ; \mathscr{L} \mid V^{\triangle})$ is zero. Thus all the non-zero intersections of the type

$$w (q ; \mathscr{L} \mid V^{\triangle}) \cdot [\boldsymbol{k} ; \mathscr{L} \mid V^{\triangle}]^*$$

have non-zero images by $\bar{\theta}^*$ and this proves the required unit multiplicity. This also completes the proof of the intersection formula.