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Enumeration of the Bounded Symmetric Domains in a Given Dimension

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Geometria. — *Enumeration of the Bounded Symmetric Domains in a Given Dimension* (*). Nota di LAWRENCE A. HARRIS, presentata (**) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Partendo dal problema della enumerazione degli oggetti in una categoria dotata di un teorema di classificazione, si costruisce una tabella dei domini limitati simmetrici olomorficamente non equivalenti di dimensione non superiore a 60. Si correggono risultati stabiliti in proposito da E. Cartan.

Our purpose is to give a table of the number of holomorphically inequivalent bounded symmetric domains in each dimension ≤ 60. Such a table has already been given by E. Cartan [3] for dimensions up to 12, but his values are incorrect because he takes the number of indecomposable domains in dimension 6 to be 5 rather than 4, the correct value [11]. (Granted this, Cartan's values for dimensions 11 and 12 are still too small by 10 and 14, respectively). By Cartan's main theorem, this problem is a special case of the problem of determining the number of objects with a given dimension in a category having a classification theorem. Specifically, let $\mathscr{C}$ be a category with a product $\times$, and call an object in $\mathscr{C}$ indecomposable if it is not isomorphic to a product of two objects in $\mathscr{C}$. Suppose that for each $\mathfrak{A} \in \mathscr{C}$ there exist indecomposable $\mathfrak{A}_1, \cdots, \mathfrak{A}_l \in \mathscr{C}$ unique up to order such that

$$\mathfrak{A} \simeq \mathfrak{A}_1 \times \cdots \times \mathfrak{A}_l ,$$

where $\simeq$ indicates isomorphism. Further, suppose dim is a function mapping $\mathscr{C}$ into the positive integers such that

$$\dim \mathfrak{A} = \dim \mathfrak{B} \qquad \text{when} \qquad \mathfrak{A} \simeq \mathfrak{B}$$

and

$$\dim (\mathfrak{A} \times \mathfrak{B}) = \dim \mathfrak{A} + \dim \mathfrak{B}$$

for any $\mathfrak{A}, \mathfrak{B} \in \mathscr{C}$. Given a positive integer n, let $r(n)$ (resp., $a(n)$) be the number of objects (resp., indecomposable objects) $\mathfrak{A}$ in $\mathscr{C}$ with $\dim \mathfrak{A} = n$. We wish to compute $r(n)$ assuming $a(k) < \infty$ for $1 \leq k \leq n$.

For example, $\mathscr{C}$ could be the category of all

A) simply connected Riemannian symmetric spaces of compact (resp., noncompact) type [5],

B) bounded symmetric domains in finite dimensional complex Euclidean spaces [3],

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C) finite dimensional Hermitian Jordan triple system [10],

D) finite dimensional semisimple Jordan pairs over **C** [9],

E) finite dimensional J*-algebras [4],

F) finite dimensional ternary algebras [6],

G) finite dimensional semisimple Lie triple systems [8],

H) finite dimensional formally real Jordan algebras over **R** [7, p. 205],

I) finite dimensional semisimple Lie algebras over **C** [5],

J) finite dimensional semisimple associative algebras over **R** [1].

TABLE I

DIM	IRR DOM	SYM DOM	DIM	IRR DOM	SYM DOM
1	1	1	31	2	190445
2	1	2	32	4	255904
3	2	4	33	3	342241
4	2	7	34	3	456862
5	2	11	35	3	607082
6	4	21	36	8	805583
7	2	31	37	2	1064489
8	3	51	38	3	1404111
9	3	80	39	3	1845784
10	5	126	40	5	2421843
11	2	187	41	2	3167299
12	4	292	42	5	4135357
13	2	427	43	2	5383039
14	3	638	44	4	6994708
15	5	935	45	6	9065141
16	5	1371	46	3	11727593
17	2	1960	47	2	15134143
18	4	2843	48	6	19498132
19	2	4024	49	3	25062080
20	4	5724	50	4	32160215
21	5	8046	51	3	41181854
22	3	11303	52	4	52648582
23	2	15687	53	2	67172985
24	5	21840	54	5	85572626
25	3	30058	55	5	108806300
26	3	41366	56	5	138138251
27	4	56525	57	3	175069880
28	6	77126	58	3	221548917
29	2	104490	59	2	279897566
30	5	141526	60	7	353116363

Let $\beta(m, k)$ be the number of different ways k objects can be chosen from m objects with replacement but without regard to the order of objects chosen. Then

$$\beta(m, k) = \binom{m + k - 1}{k}$$

and clearly

$$r(n) = \Sigma\, \beta(a(1), \pi(1)) \cdots \beta(a(n), \pi(n)), \tag{1}$$

where the sum is taken over all partitions π of n and where $\pi(k)$ denotes the number of times k occurs as a part of π. It is easy to show [2, § 1.2] that

$$\prod_{n=1}^{\infty} \frac{1}{(1 - x^n)^{a(n)}} = 1 + \sum_{n=1}^{\infty} r(n)\, x^n \tag{2}$$

when both $|x| < 1$, $\sum_{1}^{\infty} a(n)\,|x|^n < \infty$, and logarithmic differentiation of this identity [2, p. 98] yields

$$nr(n) = \Sigma\, la(l)\, r(n - kl), \tag{3}$$

where the sum is taken over all positive integers k and l satisfying $kl \leq n$. (Note that (3) holds even when one of the expressions in (2) does not converge since one may assume that $a(k) = 0$ for all $k > n$). The values of $r(n)$ can be computed recursively from (3). A general asymptotic formula for $r(n)$ is given in [2, p. 89].

Note that the values given in our table apply also to categories (C) and (D), since these categories are equivalent to category (B). The reader may obtain a table up to dimension 130 by running the following FORTRAN program on an IBM 370 computer:

```
      DOUBLE PRECISION A,S,D,R
      DIMENSION NA(130),R(131)
      N=130
      WRITE (6,1)
1     FORMAT (////6X,3HDIM,3X,7HIRR DOM,13X,7HSYM DOM/)
      DO 5 I=1,4
5     NA(I)=0
      DO 10 I=5,N
10    NA(I)=1
      NA(16)=NA(16) + 1
      NA(27)=NA(27) + 1
      NA(3)=NA(3) + 1
      NA(6)=NA(6) + 1
      DO 15 I=4,N
      K=I*(I+1)/2
      IF (K-N) 15,15,20
15    NA(K)=NA(K) + 2
20    DO 35 I=1,N
      IF (I*I-N) 25,25,40
25    DO 30 J=I,N
      K=I*J
```

```
      IF (K—N) 39,30,35
30    NA(K)=NA(K) + 1
35    CONTINUE
40    R(1)=1.0
      DO 60 N1=1,N
      S=0.0
      DO 55 J=1,N1
      DO 50 I=1,N1
      K=J*I
      IF (K — N1) 45,45,55
45    A=I*NA(I)
50    S=S + A*R(N1—K+1)
55    CONTINUE
      D=N1
      R(N1+1)=S/D
60    WRITE (6,100) N1,NA(N1),R(N1+1)
100   FORMAT (I8,I8,F24,0)
      STOP
      END
```

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