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Certain dual series equations involving Jacobi polynomials. Nota II

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Funzioni speciali. — *Certain dual series equations involving Jacobi polynomials* (*). Nota II di HARI M. SRIVASTAVA, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore dimostra che alcuni risultati ottenuti precedentemente da N. K. Thakare [*Zeitschr. Angew. Math. Mech.*, 54 (1974), 283–284] seguono facilmente da altri noti risultati.

L'Autore studia poi una generale classe di coppie di serie duali collegate ai polinomi di Jacobi.

1. INTRODUCTION

In recent years several workers have devoted considerable attention to the solutions of various pairs of dual equations involving, for instance, trigonometric series, the Fourier-Bessel series, the Fourier-Legendre series, the Dini series, and series of Jacobi and Laguerre polynomials; indeed, many of these problems arise in the investigation of certain classes of mixed boundary value problems in potential theory (cf. [6], Chapter V; *see also* [8]).

Dual series equations in which the kernels involve Jacobi polynomials of the *same* indices were first considered by Noble [4], who used Jacobi's original notation:

$$\mathfrak{F}_n(a, \lambda; \rho) = {}_2F_1[-n, a+n; \lambda; \rho], \tag{1}$$

which can be found, among other places, in the early editions of the book by Magnus and Oberhettinger [3]. On the other hand, Srivastav [7] used Szegö's notation (cf. [9], p. 62):

$$P_n^{(\alpha,\beta)}(x) = \binom{\alpha+n}{n} {}_2F_1\left[-n, \alpha+\beta+n+1; \alpha+1; \frac{1-x}{2}\right] \tag{2}$$

in order to solve a *special* case of Noble's equations. An account of both Noble's and Srivastav's solutions can be found in the recent book by Sneddon ([6], pp. 165–172), where the aforementioned connection has not been stated explicitly. {Indeed, from (1) and (2) it follows at once that

$$P_n^{(\alpha,\beta)}(1-2x) = \binom{\alpha+n}{n} \mathfrak{F}_n(\alpha+\beta+1, \alpha+1; x), \tag{3}$$

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which will evidently lead to the fact that Srivastav's equations are essentially a special case of Noble's equations (with, of course, $a = \alpha + \beta + 1$ and $\lambda = \alpha + 1$) when $\mu = \beta + \frac{1}{2}$.}

A generalization of Noble's equations (cf. [4], p. 363) was considered subsequently by Dwivedi [1], whose equations involve Jacobi polynomials of different indices in the original notation (1). By elementary changes of variables and parameters, using the relationship (3), Dwivedi's equations (1.1) and (1.2) in reference [1, p. 287] (with $a = \alpha + \beta + 1$, $\lambda = \alpha + 1$, $r = \delta + 1$, and μ replaced by $\mu + 1$) can easily be put in their *equivalent* forms:

$$(4) \qquad \sum_{n=0}^{\infty} A_n^* \frac{\Gamma(\mu + n + 1)}{\Gamma(\beta + n + 1)} P_n^{(\alpha,\beta)}(x) = f^*(x), \qquad -1 \leqq x < y,$$

$$(5) \qquad \sum_{n=0}^{\infty} A_n^* \frac{\Gamma(\alpha + \beta - \mu + n + 1)}{\Gamma(\alpha + \beta - \delta + n + 1)} P_n^{(\alpha+\beta-\delta,\delta)}(x) = g^*(x), \qquad y < x \leqq 1,$$

which would readily correspond to Noble's equations [4, p. 363] when $\delta = \beta$. Indeed, Dwivedi [1] determined the unknown sequence $\{A_n^*\}$ in terms of the prescribed functions $f^*(x)$ and $g^*(x)$ under the following alternative sets of conditions:

$$(6) \qquad \begin{cases} (i) & \alpha + \beta + 1 > \delta > \alpha > \mu > -1; \qquad \text{or} \\ (ii) & \alpha + \beta + 1 > \delta > \mu > \alpha - 1 > -2. \end{cases}$$

{Here A_n^*, f^* and g^* are suitably related to Dwivedi's A_n, f and g, respectively.}

Recently, Thakare [10] solved certain dual Jacobi series equations [using Szegö's notation (2)], which can at once be rewritten in their *equivalent* forms:

$$(7) \qquad \sum_{n=0}^{\infty} A_n^* \frac{\Gamma(\beta + \mu + n + 1)}{\Gamma(\beta + n + 1)} P_n^{(\alpha,\beta)}(x) = f^*(x), \qquad -1 \leqq x < y,$$

$$(8) \qquad \sum_{n=0}^{\infty} A_n^* \frac{\Gamma(\alpha - \mu + n + 1)}{\Gamma(\alpha + \beta - \delta + n + 1)} P_n^{(\alpha+\beta-\delta,\delta)}(x) = g^*(x), \qquad y < x \leqq 1,$$

where, in terms of the symbols used in Thakare's paper [10, p. 283, Eqs. (1) and (2)],

$$(9) \qquad \begin{cases} A_n^* = A_n / \{\Gamma(\alpha - \mu + n + 1)\, \Gamma(\beta + \mu + n + 1)\}, \\ f^*(x) = f(x)/\Gamma(\beta + 1), \qquad \text{and} \quad g^*(x) = g(x)/\Gamma(\alpha + \beta - \delta + 1). \end{cases}$$

Evidently, these last equations (7) and (8) with the parameter μ replaced *trivially* by $\mu - \beta$ are the same as the dual series equations (4) and (5), whose exact solution was given earlier by Dwivedi [1] under two alternative sets of conditions in (6) above. Thus it is easy to derive Thakare's solution (cf. [10], p. 284) from either one of Dwivedi's solutions (cf. [11), p. 289, Eq. (2.6); *see also* § 3) by merely making the aforeindicated trivial changes of variables and parameters and using the relationship (3). We omit the details involved, but remark in passing that Thakare's solvability conditions $\mu > 0$ and $\delta - \mu - \beta > 0$ are mutually contradictory in the special case $\delta = \beta$, considered by Thakare [10, p. 284, § 5], which would obviously reduce (7) and (8) to Noble's equations (and hence also to Srivastav's equations).

By applying an interesting modification of the familiar multiplying-factor technique, developed by Noble [4], we have considered (in Part I of this work) the problem of determining the unknown sequence $\{A_n\}$ satisfying the general dual series equations:

$$\sum_{n=0}^{\infty} A_n \frac{\Gamma(\mu+n+l+1)}{\Gamma(\beta+n+l+1)} P_{n+l}^{(\alpha,\beta)}(x) = f(x), \qquad -1 \leqq x < y, \tag{10}$$

$$\sum_{n=0}^{\infty} A_n \frac{\Gamma(\lambda+n+l+1)}{\Gamma(\gamma+n+l+1)} P_{n+l}^{(\gamma,\delta)}(x) = g(x), \qquad y < x \leqq 1, \tag{11}$$

where l is an arbitrary non-negative integer, $f(x)$ and $g(x)$ are prescribed functions, and, in general,

$$\min\{\alpha, \beta, \gamma, \delta, \lambda, \mu\} > -1. \tag{12}$$

We recall here the following results involving Jacobi polynomials, which were also required in the course of our earlier investigation leading to the solution (21) of (10) and (11) under the constraint (12), together with (20) and (24) below.

(*i*) The orthogonality property of the Jacobi polynomials given by [cf., e.g., [9], p.68, Eq. (4.3.3)]:

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)\,dx \tag{13}$$

$$= \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}\delta_{mn}, \qquad \alpha > -1, \quad \beta > -1,$$

where δ_{mn} is the Kronecker delta.

(*ii*) The following convenient forms of the known fractional integrals (44) and (43) on page 191 in reference [2]:

$$\int_{-1}^{\xi} (1+x)^{\beta}(\xi - x)^{\rho-1} P_n^{(\alpha,\beta)}(x)\,dx \tag{14}$$

$$= B(\beta+n+1, \rho)(1+\xi)^{\beta+\rho} P_n^{(\alpha-\rho,\beta+\rho)}(\xi), \qquad \beta > -1, \quad \rho > 0,$$

and

(15)
$$\int_{\xi}^{1} (1-x)^{\alpha} (x-\xi)^{\sigma-1} P_n^{(\alpha,\beta)}(x)\, dx$$
$$= B(\alpha+n+1, \sigma)(1-\xi)^{\alpha+\sigma} P_n^{(\alpha+\sigma,\beta-\sigma)}(\xi), \qquad \alpha > -1 \;, \; \sigma > 0,$$

where $B(\alpha, \beta)$ denotes the familiar Beta function defined by

(16)
$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad \alpha > 0 \;, \; \beta > 0 .$$

(*iii*) For integers $m \geqq 0$, we have the derivative formula

(17)
$$D_x^m \{(1-x)^{\alpha+m} P_n^{(\alpha+m,\beta-m)}(x)\}$$
$$= (-1)^m \frac{\Gamma(\alpha+m+n+1)}{\Gamma(\alpha+n+1)} (1-x)^{\alpha} P_n^{(\alpha,\beta)}(x), \qquad D_x = d/dx ,$$

which follows from the known results (7), p. 264 and (17), p. 265 in reference [5], and its complement

(18)
$$D_x^m \{(1+x)^{\beta+m} P_n^{(\alpha-m,\beta+m)}(x)\}$$
$$= \frac{\Gamma(\beta+m+n+1)}{\Gamma(\beta+n+1)} (1+x)^{\beta} P_n^{(\alpha,\beta)}(x),$$

which can be proven by using the known formulas (5), p. 264 and (17), p. 265 in reference [5].

{Incidentally, since it is fairly well known that [9, p. 59, Eq. (4.1.3)]

(19)
$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x),$$

the integral formulas (14) and (15) are essentially equivalent, and so are the derivative formulas (17) and (18).}

Indeed, by assuming that

(20)
$$\alpha + \beta = \gamma + \delta = \lambda + \mu ,$$

and by appealing to the above properties, we finally obtained (in Part I) the desired solution of the dual series equations (10) and (11) in the form:

(21)
$$A_n = \frac{(n+l)!\,(\alpha+\beta+2n+2l+1)\,\Gamma(\alpha+\beta+n+l+1)}{2^{\alpha+\beta+1}\Gamma(\lambda+n+l+1)\,\Gamma(\mu+n+l+1)}$$
$$\cdot \left[\frac{1}{\Gamma(\mu-\beta+m)} \int_{-1}^{y} (1-\xi)^{\lambda} P_{n+l}^{(\lambda,\mu)}(\xi)\, F(\xi)\, d\xi \right.$$
$$\left. + \frac{(-1)^k}{\Gamma(\lambda-\gamma+k)} \int_{y}^{1} (1+\xi)^{\mu} P_{n+l}^{(\lambda,\mu)}(\xi)\, G(\xi)\, d\xi \right],$$

where, for convenience,

$$F(\xi) = D_\xi^m \left\{ \int_{-1}^{\xi} (1+x)^\beta (\xi - x)^{\mu-\beta+m-1} f(x)\, dx \right\}, \tag{22}$$

$$G(\xi) = \int_{\xi}^{1} (x-\xi)^{\lambda-\gamma+k-1} D_x^k \{(1-x)^\gamma g(x)\}\, dx, \tag{23}$$

k, l, m, n are non-negative integers, and in addition to the parametric constraints in (12) and (20), we require that

$$\lambda > \gamma - k > -1 \quad , \quad \mu - \beta + m > 0. \tag{24}$$

In the present sequel we shall compute the values of the general series (10) and (11) on the intervals over which their values are not already prescribed.

2. Unspecified values of series (10) and (11)

The quantities of particular interest in physical applications are the values of the series (10) and (11) on the intervals where their values are not already specified. In order to determine these values, we first let

$$\sum_{n=0}^{\infty} A_n \frac{\Gamma(\mu+n+l+1)}{\Gamma(\beta+n+l+1)} P_{n+l}^{(\alpha,\beta)}(x) = \phi(x), \qquad y < x \leqq 1, \tag{25}$$

$$\sum_{n=0}^{\infty} A_n \frac{\Gamma(\lambda+n+l+1)}{\Gamma(\gamma+n+l+1)} P_{n+l}^{(\gamma,\delta)}(x) = \psi(x), \qquad -1 \leqq x < y, \tag{26}$$

where $\phi(x)$ and $\psi(x)$ are the unknown functions, l is a non-negative integer, and the various parameters are constrained as in (12), (20) and (24).

In view of the derivative formula (18), (25) can be rewritten in the form:

$$\phi(x) = (1+x)^{-\beta} \cdot D_x^r \left\{ (1+x)^{\beta+r} \sum_{n=0}^{\infty} A_n \frac{\Gamma(\mu+n+l+1)}{\Gamma(\beta+r+n+l+1)} P_{n+l}^{(\alpha-r,\beta+r)}(x) \right\}, \tag{27}$$

where r is a non-negative integer, and $y < x \leqq 1$.

Substituting for the coefficients A_n from (21) into (27), we get

$$\phi(x) = (1+x)^{-\beta} D_x^r \left\{ \frac{(1+x)^{\beta+r}}{\Gamma(\mu-\beta+m)} \int_{-1}^{y} (1-\xi)^\lambda P(x,\xi) F(\xi)\, d\xi \right. \tag{28}$$
$$\left. + \frac{(-1)^k (1+x)^{\beta+r}}{\Gamma(\lambda-\gamma+k)} \int_{y}^{1} (1+\xi)^\mu P(x,\xi) G(\xi)\, d\xi \right\}, \qquad y < x \leqq 1,$$

where, for convenience,

$$\text{(29)} \qquad P(x,\xi) = \sum_{n=0}^{\infty} \Omega_n(x,\xi) - M(x,\xi),$$

$$\text{(30)} \qquad M(x,\xi) = \sum_{n=0}^{l-1} \Omega_n(x,\xi),$$

and

$$\text{(31)} \quad \Omega_n(x,\xi) = \frac{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}{2^{\alpha+\beta+1}\Gamma(\beta+r+n+1)\Gamma(\lambda+n+1)} P_n^{(\alpha-r,\beta+r)}(x) P_n^{(\lambda,\mu)}(\xi),$$

it being understood that $M(x,\xi) = 0$ when $l = 0$.

By applying (15) and the orthogonality property (13) in (29) above, it is easily observed that

$$\text{(32)} \quad \sum_{n=0}^{\infty} \Omega_n(x,\xi) = \frac{(x-\xi)^{\lambda-\alpha+r-1}}{\Gamma(\lambda-\alpha+r)} (1+x)^{-\beta-r}(1-\xi)^{-\lambda} H(x-\xi),$$

where $\Omega_n(x,\xi)$ is given (31), $H(t)$ denotes Heaviside's unit function, and

$$\text{(33)} \qquad \lambda > \alpha - r > -1 .$$

Substituting from (32) into (29), and then into (28), we finally have

$$\text{(34)} \quad \phi(x) = \frac{(1+x)^{-\beta}}{\Gamma(\lambda-\alpha+r)} D_x^r \left\{ \frac{1}{\Gamma(\mu-\beta+m)} \int_{-1}^{y} (x-\xi)^{\lambda-\alpha+r-1} F(\xi)\, d\xi \right.$$

$$\left. + \frac{(-1)^k}{\Gamma(\lambda-\gamma+k)} \int_{y}^{x} (1-\xi)^{-\lambda}(1+\xi)^{\mu}(x-\xi)^{\lambda-\alpha+r-1} G(\xi)\, d\xi \right\}$$

$$- (1+x)^{-\beta} D_x^r \left\{ \frac{(1+x)^{\beta+r}}{\Gamma(\mu-\beta+m)} \int_{-1}^{y} (1-\xi)^{\lambda} M(x,\xi) F(\xi)\, d\xi \right.$$

$$\left. + \frac{(-1)^k (1+x)^{\beta+r}}{\Gamma(\lambda-\gamma+k)} \int_{y}^{1} (1+\xi)^{\mu} M(x,\xi) G(\xi)\, d\xi \right\}, \qquad y < x \leqq 1,$$

where $F(\xi)$, $G(\xi)$ and $M(x,\xi)$ are given by (22), (23) and (30), respectively, and the parametric restrictions in (12), (20), (28) and (33) are assumed to hold, r being a non-negative integer.

In a similar manner, by applying (13), (14) and (17) to our hypothesis (26) we can readily show that

$$\text{(35)} \qquad \psi(x) = \frac{(1-x)^{-\gamma}}{\Gamma(\lambda-\gamma+s)} (-D_x)^s \left\{ \frac{1}{\Gamma(\mu-\beta+m)} \right.$$

$$\cdot \int_{x}^{y} (1-\xi)^{\lambda}(1+\xi)^{-\mu}(\xi-x)^{\gamma-\lambda+s-1} F(\xi)\, d\xi +$$

$$+ \frac{(-1)^k}{\Gamma(\lambda-\gamma+k)} \int_y^1 (\xi - x)^{\gamma-\lambda+s-1}\, G(\xi)\, d\xi \Big\}$$

$$-(1-x)^{-\gamma}(-D_x)^s \Big\{ \frac{(1-x)^{\gamma+s}}{\Gamma(\mu-\beta+m)} \int_{-1}^{y} (1-\xi)^{\gamma}\, N(x,\xi)\, F(\xi)\, d\xi$$

$$+ \frac{(-1)^k (1-x)^{\gamma+s}}{\Gamma(\lambda-\gamma+k)} \int_y^1 (1+\xi)^{\mu}\, N(x,\xi)\, G(\xi)\, d\xi \Big\}, \qquad -1 \leqq x < y,$$

where $F(\xi)$ and $G(\xi)$ are given by (22) and (23), respectively,

$$N(x,\xi) = \sum_{n=0}^{l-1} \Lambda_n(x,\xi), \tag{36}$$

$$\Lambda_n(x,\xi) = \frac{n!\,(\alpha+\beta+2n+1)\,\Gamma(\alpha+\beta+n+1)}{2^{\alpha+\beta+1}\,\Gamma(\mu+n+1)\,\Gamma(\gamma+n+s+1)}\, P_n^{(\gamma+s,\delta-s)}(x)\, P_n^{(\lambda,\mu)}(\xi), \tag{37}$$

with $N(x,\xi) = 0$ if $l = 0$, and in addition to the parametric constraints given by (12), (20) and (24), we must have

$$\gamma - \lambda + s > 0 \quad , \quad \delta - s > -1, \tag{38}$$

s being a non-negative integer.

3. Concluding remarks

Obviously, when $l = 0$, $\lambda = \alpha + \beta - \mu$ and $\gamma = \alpha + \beta - \delta$, the dual series equations (10) and (11) would correspond to Dwivedi's equations (4) and (5), and hence also to Thakare's equations (7) and (8) if we replace the parameter μ by $\mu + \beta$. Thus, under the special cases just stated, our solution given by equation (21) would readily yield the results obtained by these earlier writers. And indeed, it will lead to the solution of the dual series equations considered by Noble [4] when we further set $\delta = \beta$, and to that of Srivastav's equations [7] if, in addition to the aforementioned parametric constraints, we set $\mu = \beta + 1/2$.

We should like to conclude by remarking further that (by appropriately specializing our equations (34) and (35) of the preceding section) we can easily obtain the values of the dual series, considered by Dwivedi [1] and Thakare [10], on the intervals over which their values are not already specified. As a matter of fact, this important aspect of the analysis presented here was not given by either of these earlier writers.

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