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Self-distributive infra-near rings

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Algebra. — *Self-distributive infra-near rings.* Nota di MIRELA STEFĂNESCU, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — Si studiano le proprietà di una particolare struttura algebrica che soddisfa le leggi di auto-distributivitá.

The following question has its own interest and its own history in the theory of algebraic systems: Which are the properties of an algebraic system S having a binary operation—we denote it multiplicatively—that satisfies the self-distributivity laws:

$$\text{(I)} \qquad x\cdot(y\cdot z) = (x\cdot y)\cdot(x\cdot z)\,;$$

$$\text{(II)} \qquad (x\cdot y)\cdot z = (x\cdot z)\cdot(y\cdot z)\,, \qquad \text{for all} \quad x\,,y\,,z \in \mathrm{S}\,.$$

For quasigroups, these laws had been studied by Bruck [1], in connexion with so called "the alternation law":

$$\text{(III)} \qquad (x\cdot y)\cdot(z\cdot w) = (x\cdot z)\cdot(y\cdot w)\,, \qquad \text{for all} \quad x\,,y\,,z\,,w \in \mathrm{S}.$$

J. Ruedin [6, 7] had studied them for groupoids, M. Petrich [5], for semigroups and rings, C. Ferrero-Cotti [3], for near-rings, Z. Fiedorowicz [4], for linear algebras over fields.

We study here the self-distributive infra-near rings. The notion of infra-near ring was introduced by the author [8], in connexion with a generalization of the distributivity laws satisfied by the multiplication over addition in rings and near-rings. The set of affine transformations of a linear space over a field is a right near-ring and a *left infra-near ring* with respect to the pointwise addition and mapping composition, and there are some interesting properties of this infra-near ring (studied, as an infra-near ring, and generalized by the author [10]). A weak ring (see Climescu [2]) is also a left and right infra-near ring with commutative addition. Comments about the now notion, many examples, ways to find infra-distributive multiplaccation on a group, the study of the structure of special infra-near rings made the aims of papers [8, 9, 10].

1. Recall some definitions and properties in the theory of infra-near rings.

(*) Nella seduta del 12 gennaio 1980.

(1.1) DEFINITION. A *left infra-near ring* (*left IN-ring*) is a triple $(S, +, \cdot)$, where $(S, +)$ is a group, $(S, \cdot)$ is a semigroup and the multiplication is left infra-distributive with respect to the addition, i.e. it satisfies the following law:

$$x\cdot(y+z) = x\cdot y - x\cdot 0 + x\cdot z\,, \qquad \text{for all } x, y, z \in S\,. \tag{1}$$

If, instead of (1), the right infra-distributivity law is satisfied, i.e.:

$$(x+y)\cdot z = x\cdot z - 0\cdot z + y\cdot z\,, \qquad \text{for all } x, y, z \in S\,, \tag{2}$$

then the triple $(S, +, \cdot)$ is called a *right infra-near ring* (*right IN-ring*).

In a left IN–ring, we have:

$$x\cdot(-y) = x\cdot 0 - x\cdot u + x\cdot 0\,, \qquad \text{for all } x, y \in S\,. \tag{3}$$

Generally, $x\cdot 0 \neq 0$, for an element x of a left IN-ring S, even if $x = 0$. If $x\cdot 0 = 0$, then x is called *left distributive*. If $x\cdot 0 = x$, then x is called *left weakly distributive*. If $x\cdot 0 = 0$ (resp. $0\cdot x = 0$), for all x in the left IN-ring (resp. the right IN-ring) S, then it is a *left near-ring* (resp. a *right near-ring*). If $x\cdot y = x\cdot 0$, for all $x, y \in S$, then $(S, +, \cdot)$ is called a *trivial left IN-ring*.

(1.2) DEFINITION. A nonvoid subset B of a left IN-ring S is called a *left ideal of* S, if $(B, +)$ is a normal subgroup of $(S, +)$ and for all $x \in S$ and $b \in B$, whe have:

$$x\cdot b - x\cdot 0 \in B. \tag{4}$$

If $(B, +)$ is a normal subgroup of $(S, +)$, for which:

$$(b+x)\cdot y - x\cdot y \in B\,, \qquad \text{for all } x, y \in S\,, \quad b \in B\,, \tag{5}$$

then B is called a *right ideal of* S. If S is a left IN-ring and a right near-ring, then (5) becomes the usual condition for right ideals in rings. A left and right ideal is called an *ideal of* S.

The ideals of S are just the kernels of IN-ring homomorphisms from S to another left IN-ring. The subset of the left IN-ring S, given by:

$$\mathrm{Ann}_r(x) = \{t \mid t \in S\,, x\cdot t = x\cdot 0\}\,, \tag{6}$$

for an element $x \in S$, is called the *right annihilator of* x. The intersection of all $\mathrm{Ann}_r(x)\,, x \in S$, *is the right annihilator of* S:

$$\mathrm{Ann}_r(S) = \{t \mid t \in S\,, x\cdot t = x\cdot 0\,, \forall x \in S\}\,, \tag{7}$$

and it is a left ideal in S. If S is also a right near-ring, $\mathrm{Ann}_r(S)$ is an ideal, while $\mathrm{Ann}_r(x)$ is a right ideal of S, for each $x \in S$.

2. We shall study now the structure of left IN-rings having a self-distributive multiplication.

(2.1) DEFINITION. A left IN-ring $(S, +, \cdot)$ is called *self-distributive*, if its multiplication satisfies the laws (I) and (II).

Everywhere in the paper, we shall denote by S *a self-distributive IN-ring.*

(2.2) *Remark.* If e is a left distributive element of S, then the mapping $\varphi_e : S \to S$, defined by $\varphi_e(x) = e \cdot x$, for all $x \in S$, is an endomorphism of S, whose kernel, $\mathrm{Ann}_r(e)$, is an ideal of S.

(2.3) *Remark.* If we call, as usually, *left cancellable* a nonzero element $c \in S$, such that:

$$c \cdot x = c \cdot y , \quad \text{for } x, y \in S , \quad \text{implies } x = y , \tag{8}$$

then a necessary and sufficient condition for c to be left cancellable is:

$$c \cdot x = c \cdot 0 , \quad \text{for } x \in S , \quad \text{implies } x = 0 . \tag{9}$$

The condition (9) is obviously necessary. It is also sufficient. Indeed, if $c \cdot x = c \cdot y$, then $c \cdot (x - y) = c \cdot x - c \cdot 0 + (c \cdot 0 - c \cdot y + c \cdot 0) = c \cdot 0$, and by virtu of (9), we have: $x - y = 0$, hence $x = y$.

(2.4) *Remark.* Since the multiplication of S is associative, the set of all left cancellable elements of S is a multiplicative subsemigroup of $(S, \cdot)$. Moreover we can prove the following:

(2.5) LEMMA. *If the self-distributive IN-ring* S *has a left cancellable element e, then e is a left identity of S and all the elements of* S *are idempotent.*

Proof. For all $x \in S$, we have: $(e \cdot e) \cdot x = e \cdot (e \cdot x) = (e \cdot e) \cdot (e \cdot x)$, hence $x = e \cdot x$, since $e \cdot e$ is left cancellable. Therefore e is a left identity and $e \cdot e \mp e^2 = e$. Now $x^2 = x \cdot x = (e \cdot x) \cdot (e \cdot x) = (e \cdot e) \cdot x = e \cdot x = x$, for all $x \in S$, i.e. each x is idempotent.

(2.6) *Remark.* ([5], Prop. 2). In a self-distributive semigroup $(S, \cdot)$, the elements of the form $x \cdot y \cdot z$, with $x, y, z \in S$, are idempotent.

We can prove also, by induction on n:

$$x^n = x^3 , \quad \text{for all } x \in S \quad \text{and } n \geq 3 ; \tag{10}$$

$$(x \cdot y)^n = x^n \cdot y^n = x^2 \cdot y^2 = x \cdot y^2 = x^2 \cdot y , \quad \text{for all } x, y \in S , \ n \geq 2 ; \tag{11}$$

$$(x \cdot y) \cdot z = (x \cdot y) \cdot z^n = x^n \cdot (yz) , \quad \text{for all } x, y, z \in S , \ n \geq 1 . \tag{12}$$

(2.7) *Remark.* A trivial left IN-ring S is self-distributive, since the identity $x \cdot y = x \cdot 0$ implies $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z) = (x \cdot y) \cdot z = (x \cdot z) \cdot (y \cdot z) = x \cdot 0$, for all $x, y, z \in S$. Its elements of the form $x \cdot 0$ are idempotent. If $x \cdot 0 = x$, for all $x \in S$, then each $x \in S$ is, obviously, a left zero: $x \cdot y = x \cdot 0 = x$, for all $x, y \in S$, and $x^2 = x$.

An example of self-distributive left IN-ring without this property is $S = R^5$, where R is a ring, endowed with binary operations:

$$x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4, x_5 + x_2 \cdot y_3 + x_1 \cdot y_4 + y_5),$$

$$x \cdot y = (0, 0, x_3, x_4, x_5).$$

(2.8) DEFINITION. An element z in S is called *quasi-nilpotent*, if there exists a natural number n, such that $x \cdot z^n = x \cdot 0$, for all $x \in S$, i.e. $z^n \in \mathrm{Ann}_r(S)$.

Obviously, if $z \in S$ is nilpotent, i.e. there exists n, such that $z^n = 0$, then z is quasi-nilpotent, while the converse is not always true. If S is a left near-ring, the two notions are not different.

(2.9) *Remark.* By virtue of (10), in order to check if $z \in S$ is quasi-nilpotent, we have to verify if z, z^2 or z^3 belong to $\mathrm{Ann}_r(S)$. Moreover if $0 \cdot 0 = 0$ in S, then, by (12), $x^m \cdot z^n = x \cdot z^2$, when either m or n is strictly greater then 1. Therefore we must only verify whether z, z^2, z^3 are in $\mathrm{Ann}_r(x)$, for those x of S which are not powers of another element of S.

By using (12), we have immediately:

(2.10) *Remark.* For any quasi-nilpotent element z of the self-distributive left IN-ring S, and for any $x, y \in S$, the following equality holds:

(13) $$(x \cdot y) \cdot z = (x \cdot y) \cdot 0.$$

The last remark helps us to prove the following:

(2.11) PROPOSITION. *A self-distributive left IN-ring* S *is without propre right annihilators if and only if it satisfies, disjunctively, one of the two conditions*:

(i) *It is a trivial self-distributive left IN-ring, eventually composed by left zeros*;

(ii) *All the elements of* S, *with one possible exception* (*which is a left zero*), *are left identities.*

Proof. The sufficience is pretty obvious: (i) by (2.7), (ii) by straightforward calculations. To prove the necessity, let S be a non-trivial self-distributive left IN-ring without proper right annihilators. Then there exist $a, x_0 \in S$, $x_0 \neq 0$, such that $a \cdot x_0 \neq a \cdot 0$, hence $x_0 \notin \mathrm{Ann}_r(a)$, $\mathrm{Ann}_r(a) = 0$ and a is a left cancellable element of S, by virtue of (2.3). By (2.5), a is a left identity of S and each element of S is idempotent. For an element $b \in S$, $b \neq 0$, we deduce, from the disjuntive situations: $b^2 = b = b \cdot 0$ or $b^2 = b \neq \neq b \cdot 0$, than b is a left zero (because of the fact that $b \in \mathrm{Ann}_r(b) = S$) or a left identity (because of the fact that $b \notin \mathrm{Ann}_r(b) = 0$). For $b = 0$, we may also have one of these two situations, hence 0 could be a left identity in S or a left zero. Now, if there exists a left identity, a, we could not have two distinct left zeros, z, z_1 in S, since we obtain a contradiction: $a \cdot (z \cdot z_1) = = z \cdot z_1 = z \neq z_1 = z_1 \cdot z = (a \cdot z_1) \cdot (z \cdot z_1) = (a \cdot z) \cdot z_1 = a \cdot (z \cdot z_1)$.

(2.12) *Remark.* If S is in the case (ii) of (2.11), with all the elements left identities, then it is a trivial left near-ring, with the multiplication: $x \cdot y = y$, for all $x, y \in S$. This is the case obtained for self-distributive near-rings by Ferrero–Cotti ([3], Teorema 6). If $b \neq 0$ is the left zero in the same case (ii) of (2.11), then $(S, +, \cdot)$ is isomorphic—as a left IN-ring—with $(S, +_b, *)$, where:

$$x +_b y = x + b + y, \qquad \text{for all} \quad x, y \in S;$$

$$x * y = y, \qquad \text{for all} \quad x \in S \setminus \{0\} \qquad \text{and all} \quad y \in S;$$

$$0 * y = 0, \qquad \text{for all} \quad y \in S,$$

the izomorphism being given by $\rho(x) = x + b$, for all $x \in S$, from $(S, +_b, *)$ onto $(S, +, \cdot)$.

Here we have some examples of self-distributive left IN-rings:

(2.13) *Example.* $T_c(G)$, the set of all constant functions on a group $(G, +)$, is a trivial self-distributive left IN-ring with all elements weakly left distributive, with respect to pointwise addition and mappings composition.

(2.14) *Example.* Let $(G, +)$ be a group. Then $S = G \times G$, with the operations:

$$x + y = (x_1 + y_1, x_2 + y_2),$$

$$x \cdot y = (x_1, y_2), \qquad \text{for all} \quad x, y \in S,$$

is a nontrivial self-distributive left IN-ring with proper right annihilators, since $\mathrm{Ann}_r(S) = \{(y_1, 0 \mid u_1 \in G\}$ and all the elements of S are idempotent.

(2.15) *Example.* Let $(R, +, \cdot)$ be a ring; then $S = R^3$, with the operations:

$$x + y = (x_1 + y_1, x_2 + y_2, x_3 + x_2 \cdot y_1 + y_3),$$

$$x \cdot y = (y_1, 0, x_3), \qquad \text{for all} \quad x, y \in S,$$

is a nontrivial self-distributive left IN-ring with proper annihilators, because $\mathrm{Ann}_r(S) = \{(0, y_2, y_3) \mid y_2, y_3 \in R\}$. The elements of the form $(x_1, 0, x_3)$, hence the elements $x \cdot y$, with $x, y \in S$, are idempotent.

(2.16) COROLLARY. *In a self-distributive left IN-ring* S *with* $0 \cdot x = 0$, *for all* $x \in S$, *a minimal nontrivial right ideal* B *is composed by only left nonzero identities, hence* B *is a trivial left near-ring.*

Indeed, such an ideal B does not have proper right annihilators, because of its minimality, and it has a left identity, because it is not trivial. We can apply (2.11), (ii), to B.

(2.17) COROLLARY. *If a self-distributive left IN-ring* S, *with* $0 \cdot x = 0$, *for all* $x \in S$, *is a direct sum of minimal nontrivial right ideals, then it is a semisimple left near-ring with self-distributive multiplication.*

The corollary is obtained immediately, by using (2.16) and Teorema 7, [3].

(2.18) PROPOSITION. *Let* S *be a self-distributive left IN-ring which is also a right near-ring. Then*:

(i) $D = \{d \mid d \in D, d \cdot 0 = 0\}$ *is a distributive near-ring with self-distributive multiplication*;

(ii) $W = \{w \mid w \in S, w \cdot 0 = w\}$ *is an IN-ring composed only by left zeros of* S;

(iii) $S = D + W = W + D$, *and the multiplication is* $(d_1 + w_1) \cdot \cdot (d_2 + w_2) = d_1 \cdot d_2 + w_1$, *for all* $d_i + w_i \in S$, $i = 1, 2$, *and* $D \cdot W = 0$.

Proof. (i) and (ii) can be easily verified. To prove (iii), let us note that, for any $x \in S$, $x = (x - x \cdot 0) + x \cdot 0 = x \cdot 0 + (-x \cdot 0 + x)$, with $x - x \cdot 0$ and $-x \cdot 0 + x$ in D and $x \cdot 0$ in W. Moreover, for each $d \in D$ and $w \in W$, $d \cdot w = d \cdot (w \cdot 0) = (d \cdot w) \cdot 0 = (d \cdot 0) \cdot (w \cdot 0) = 0 \cdot w = 0$, hence $D \cdot W = 0$. Analogously, $W \cdot S = W$. We can easily see that d and w in the expression of $x \in S$ are unique. Then $x_1 \cdot x_2 = (d_1 + w_1) \cdot (d_2 + w_2) = d_1 \cdot (d_2 + w_2) + w_1 \cdot (d_2 + w_2) = d_1 \cdot d_2 + d_1 \cdot w_2 + w_1 = d_1 \cdot d_2 + w_1$, since $d_1 \cdot w_2 = 0$.

We use the above notations in the following comments.

(2.19) PROPOSITION. *The only self-distributive left IN-rings which are also right near-rings with identity (left identity) are the Boole rings (the left and right near-rings with self-distributive multiplication).*

Proof. Indeed, if $e \in S$ is a left identity, then $e \cdot 0 = 0$ and $e \in D$, by (2.18), (i). But $e \cdot w = w \in D \cdot W = 0$, hence $W = 0$, and $S = D$ is a left near-ring. If e is the identity of S, then S is a left and right unitary self-distributive near-ring, therefore S is a Boole ring (see [3], Osservazione 8).

In the second case of (2.19), the structure of the near-ring is given by the theorems 9 and 10 from [3] and in the theorem 2, from [5].

The two lemmas and the theorem that follow give the structure of some special self-distributive left IN-rings.

(2.20) LEMMA. *If* S *is a self-distributive left IN-ring which is also a right near-ring, then* $J = [D, D]$, *the commutator subgroup of* $(D, +)$, *is an ideal of* S, *with* $J^2 = 0$, $J \cdot S = 0$.

Proof. J is a normal subgroup in D, and $w + [d_1, d_2] - w = [w + d_1 - w, w + d_2 - w] \in J$, $d + j - d \in J$, hence $x + j - x \in J$, for all $x \in S, j \in J, w \in W, d_1, d_2 \in D$. By calculating, we obtain $D \cdot J = J \cdot D = J \cdot W = 0$, and then $J \cdot S = 0$, $J^2 = 0$. If $x = d + w \in S$ and $j \in J$, then $x \cdot j - x \cdot 0 = d \cdot j + w - w = d \cdot j = 0 \in J$ and $j \cdot x = 0 \in J$, hence J is an ideal of S.

(2.21) *Remark.* As D is a left and right near-ring with self-distributive multiplication, by applying the theorems 9, [3] and 2, [5], we find that D/J, which is a ring, is equal to B + R, where B is a Boole ring and R is a ring with $R^3 = 0$.

But in the hypotheses of (2.20), there exists S/J, and each class of S/J either constains only one element of W or does not contain any element of W. Then the subset $\overline{W} = \{w + J \mid w \in W\}$ of S/J is a self-distributive left IN-ring of left zeros isomorphic to W, and $S/J = D/J + \overline{W}$.

(2.22) LEMMA. *Under the same hypotheses for* S *as in* (2.20), *is* $S = D \oplus W$ (*direct sum of groups*), *then each ideal* B *of* D *is an ideal of* S *and* S *has an ideal* T *with* $T^3 = 0$.

Proof. Because of the property of the direct sum of groups, B is a normal subgroup in (S , +). Let $x = d + w \in S$, $b \in B$ be arbitrary elements. Then $x \cdot b - x \cdot 0 = d \cdot b + w - w = d \cdot b \in B$ and $b \cdot x = b \cdot d \in B$. The theorem 9 from [3] gives the existence of an ideal of D, let us denote it by T, with $T^3 = 0$, namely $\eta^{-1}(R)$, where $\eta : D \to D/J$ is the natural epimorphism and R is the ring with $R^3 = 0$ from the decomposition of D/J in (2.21). It is known that, in the hypotheses of (2.20), $B = D^3$ is a Boole ring (Teorema 11, [3]).

(2.23) THEOREM. *If* S *is a self-distributive left IN-ring which is also a right near-ring and* $S = D \oplus W$ (*as groups*), *then* $S = (B + T) \oplus W$, *where* B *is a Boole ring,* T *is an ideal of* S *with* $T^3 = 0$, $T \subseteq D$, $B \cap T = 0$, $B \cdot T = 0$ *and* W *is a trivial IN-ring containing all the left zeros of* S.

The proof results from the above lemmas and comments.

If we give up the associativity law for the multiplication of S, then there are some results for a self-distributive nonassociateve left IN-ring that can be obtained as their analogs for linear nonassociative algebras over fields (see [4]). For example:

$$x^2 \cdot x = x \cdot x^2 = x^2 \cdot x^2 = x^3 , \qquad \text{for all } x \in S ,$$

x^2 is idempotent (see also [7], Prop. 1),

(S , ·) is power-associative, i.e. by induction on m and n, one can show that: $x^m = x^3$, for $m \geq 3$, where $x^m = x^{m-1} \cdot x$; $x^m \cdot x^2 = x^{m+2} = x^3$, for $m \geq 1$; $x^m \cdot x^n = x^{m+n}$, for all $x \in S$ and all natural numbers m and n (see [4], Theorems 5, 6).

Because in a left and right nonassociative near-ring S the multiplication satisfies the law: $x \cdot y + z \cdot w = z \cdot w + x \cdot y$, for any $x , y , z , w \in S$, if S is self-distributive, then, for all $x , y , u , v \in S$, the following equalities hold:

$$(v \cdot x) \cdot (u \cdot y) = -(u \cdot x) \cdot (v \cdot y) \, ; \, (v \cdot u) \cdot x = -(u \cdot v) \cdot x \, ;$$

$$(x \cdot u) \cdot (y \cdot v) = -(x \cdot v) \cdot (y \cdot u) \, ; \, x \cdot (u \cdot v) = -x \cdot (v \cdot u) \, ;$$

$$(x \cdot u) \cdot (v \cdot y) = (u \cdot v) \cdot (x \cdot y) \, ;$$

$$(x \cdot u) \cdot (v \cdot y) = -(x \cdot v) \cdot (u \cdot y) .$$

If S has a right identity r, then:

$$v \cdot u = - u \cdot v \,, \qquad \text{for all} \quad u \,, v \in \mathrm{S} \,.$$

If S has an identity e, then, for all $u \,, v \,, x \in \mathrm{S}$:

$$u \cdot v = v \cdot u \qquad \text{and} \quad x + x = 0 .$$

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