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Samuel A. Ilori

A Characterisation of Symmetric Functions

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

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Presiede il Presidente della Classe ANTONIO CARRELLI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *A Characterisation of Symmetric Functions.* Nota di SAMUEL A. ILORI, presentata (*), dal Socio E. MARTINELLI.

RIASSUNTO. — Si esprimono le funzioni simmetriche, elementari e complete, mediante polinomi inscatolati (« nested polynomials »). Ciò permette di esprimere i polinomi di Poincaré e il genere di Todd generalizzato delle varietà di bandiere per mezzo di funzioni simmetriche.

1. INTRODUCTION

Consider the identity

$$\prod_{i=0}^{q} (1 + \gamma_i t) = \sum_{r=0}^{q+1} \sigma_r (\gamma_0, \cdots, \gamma_q) t^r.$$

Then for $r = 1, \cdots, q+1$, $\sigma_r = \sigma_r (\gamma_0, \cdots, \gamma_q)$ is called the r-th elementary symmetric function in $\gamma_0, \cdots, \gamma_q$ $(\sigma_0 = 1)$. σ_0 is the sum of all monomials in $\gamma_0, \cdots, \gamma_q$ of total degree r.

Consider also the identity

$$\prod_{i=0}^{q} (1 - \gamma_i t)^{-1} = \sum_{r=0}^{\infty} \overline{\sigma}_r (\gamma_0, \cdots, \gamma_q) t^r.$$

Then for $r = 1, 2, 3, \cdots$, $\overline{\sigma}_r = \overline{\sigma}_r (\gamma_0, \cdots, \gamma_q)$ is called the r-th complete symmetric function in $\gamma_0, \cdots, \gamma_q$ $(\overline{\sigma}_0 = 1)$

$$\overline{\sigma}_r = \Sigma \gamma_0^{k_0}, \cdots, \gamma_q^{k_q},$$

(*) Nella seduta del 12 gennaio 1980.

where the summation is over all permutations $(k_0, \cdots, k_q)$ satisfying the conditions $0 \leq k_i \leq r$ $(0 \leq i \leq q)$ and $\Sigma k_i = r$.

In section 2, we shall give an explicit characterisation of σ_r and $\overline{\sigma}_r$ in terms of nested polynomials. This characterisation then enables us to show that the Poincaré polynomial (cf. Section 26 of [1]) of a Grassmannian, $G_{m+1}(\mathbf{C}^{n+1})$, can be expressed as a complete symmetric function. Similarly, one shows that the generalised Todd genus (cf. Section 10 of [3]) of an incomplete flag manifold,

$$W(a_1, \cdots, a_{m-1}, n) = U(n+1) / U(a_1+1) \times U(a_2 - a_1) \times \cdots \times U(n - a_{m-1}),$$

can be expressed as a product of complete symmetric functions.

Notation. ([2]). We shall interpret the expression

$$\sum_{i_1=0}^{n} \gamma_{i_1} \cdot \sum_{i^0=0}^{1_1} \gamma_{i_0}$$

to mean

$$\gamma_0^2 + \gamma_1 \cdot \sum_{i_0=0}^{1} \gamma_{i_0} + \gamma_2 \cdot \sum_{i_0=0}^{2} \gamma_{i_0} + \cdots + \gamma_n \cdot \sum_{i_0=0}^{n} \gamma_{i_0} .$$

By iteration, the expression

$$\sum_{i_k=0}^{n} \gamma_{i_k} \cdot \sum_{i_{k-1}=0}^{i_k} \gamma_{i_{k-1}} \cdots \sum_{i_0=0}^{i_1} \gamma_{i_0}$$

will be interpreted similarly. We shall denote the last expression by

$$\prod_{i_k=0}^{n} \sum_{i_j=0}^{i_{j+1}} \gamma_{i_j}, \qquad \text{where} \quad i_{k+1} = n,$$

and it is a k-nested polynomial in $\gamma_0, \cdots, \gamma_n$.

2. Symmetric functions

PROPOSITION 2.1. *The elementary symmetric function* $\sigma_r = \sigma_r(\gamma_0, \cdots, \gamma_q)$, $r = 1, \cdots, q+1$, *in* $\gamma_0, \cdots, \gamma_q$ *is given by*

$$\sigma_r(\gamma_0, \cdots, \gamma_q) = \prod_{j=-1}^{r-2} \sum_{i_j=1}^{i_{j+1}} \gamma_{i_j+1},$$

where $i_{r-1} = q - r + 2$.

Proof. The proof will be by induction on r. When $r = 1$, the Proposition is true since

$$\sigma_1(\gamma_0, \cdots, \gamma_q) = \sum_{i_{-1}=1}^{q+1} \gamma_{i_{-1}-1} = \gamma_0 + \cdots + \gamma_q .$$

Now assume that

$$\sigma_k(\gamma_0, \cdots, \gamma_q) = \prod_{j=-1}^{k-2} \sum_{i_j=1}^{i_{j+1}} \gamma_{i_j+j}, \qquad \text{where} \quad i_{k-1} = q - k + 2$$

for all $k < r$. We then show that the Proposition is true for $k = r$. Now

$$\begin{aligned} \sigma_r &= \sum_{i_s=1}^{q-r+2} \gamma_{i_s+r-2}\, \sigma_{r-1}(\gamma_0, \cdots, \gamma_{i_s+r-3}) \\ &= \sum_{i_s=0}^{q-r+2} \gamma_{i_s+r-2} \left[\sum_{j=-1}^{r-3} \prod_{i_j=1}^{i_{j+1}} \gamma_{i_j+j} \right], \qquad \text{where} \quad i_{r-2} = i_s \\ &\qquad\qquad \text{(by the inductive hypothesis)} \\ &= \prod_{j=-1}^{r-2} \sum_{i_j=1}^{i_{j+1}} \gamma_{i_j+j}, \qquad \text{where} \quad i_{r-1} = q - r + 2 . \end{aligned}$$

Hence the Proposition is true for all r.

PROPOSITION 2.2. *The complete symmetric function* $\overline{\sigma}_r = \overline{\sigma}_r(\gamma_0, \cdots, \gamma_q)$, $r = 1, 2, 3, \cdots$ *in* $\gamma_0, \cdots, \gamma_q$ *is given by*

$$\overline{\sigma}_r(\gamma_0, \cdots, \gamma_q) = \prod_{j=0}^{r-1} \sum_{i_j=0}^{i_{j+1}} \gamma_{i_j},$$

where $i_r = q$.

Proof. The proof will also be by induction on r. When $r = 1$, the Proposition is true since

$$\overline{\sigma}_1(\gamma_0, \cdots, \gamma_q) = \sum_{i_j=0}^{q} \gamma_{i_j} = \gamma_0 + \cdots + \gamma_q .$$

Now assume that

$$\overline{\sigma}_k(\gamma_0, \cdots, \gamma_q) = \prod_{j=0}^{k-1} \sum_{i_j=0}^{i_{j+1}} \gamma_{i_j}, \qquad \text{where} \quad i_k = q ,$$

for all $k < r$. We then show that the Proposition is true for $k = r$. Now

$$\begin{aligned} \overline{\sigma}_r &= \sum_{s=0}^{q} \gamma_s\, \overline{\sigma}_{r-1}(\gamma_0, \cdots, \gamma_s) \\ &= \sum_{s=0}^{q} \gamma_s \left[\prod_{j=0}^{r-2} \sum_{i_j=0}^{i_{j+1}} \gamma_{i_j} \right], \qquad \text{where} \quad i_{r-1} = s \\ &\qquad\qquad \text{(by the inductive hypothesis)} \\ &= \prod_{j=0}^{r-1} \sum_{i_j=0}^{i_{j+1}} \gamma_{i_j}, \qquad \text{where} \quad i_r = q . \end{aligned}$$

Hence the Proposition is true for all r.

COROLLARY 2.3.

$$\frac{(1-t^{2(m+2)})(1-t^{2(m+3)})\cdots(1-t^{2(n+1)})}{(1-t^2)(1-t^4)\cdots(1-t^{2(n-m)})} = \overline{\sigma}_{m+1}(1, t^2, t^4, \cdots, t^{2(n-m)}).$$

Proof. The proof follows easily from the Proposition and Lemma 2.1 in [2].

COROLLARY 2.4. *The Poincaré polynomial of a Grassmannian*, $G_{m+1}(\mathbf{C}^{n+1})$, *is given by*

$$P(G_{m+1}(\mathbf{C}^{n+1}), t) = \overline{\sigma}_{m+1}(1, t^2, t^4, \cdots, t^{2(n-m)}).$$

Proof. The proof follows easily from Corollary 2.3 above and Corollary 2.2 in [2], where $W(m, n) = G_{m+1}(\mathbf{C}^{n+1})$.

COROLLARY 2.5. *The generalised Todd genus of an incomplete flag manifold.*

$$W(a_1, \cdots, a_{m-1}, n) = U(n+1) / U(a_1+1) \times U(a_2-a_1) \times \cdots \times U(n-a_{m-1}),$$

is given by

$$T_y(W(a_1, \cdots, a_{m-1}, n)) =$$

$$= \prod_{k=1}^{m-1} \overline{\sigma}_{a_k-a_{k-1}}(1, -y, \cdots, (-y)^i, \cdots, (-y)^{n-a_k}) \quad , \quad (a_0 = -1).$$

Proof. The proof follows easily from Proposition 2.1 above and Theorem 2.3 in [2].

REFERENCES

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