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Rings whose finitely generated torsion modules in the sense of Dickson decompose into direct sums of cyclic submodules

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Algebra. — *Rings whose finitely generated torsion modules in the sense of Dickson decompose into direct sums of cyclic submodules* (*). Nota di ALBERTO FACCHINI (**), presentata (***) dal Corrisp. I. BARSOTTI.

RIASSUNTO. — Si studia la struttura dei moduli di torsione secondo Dickson sui *d*-anelli, ossia sugli anelli commutativi per i quali ogni modulo finitamente generato di torsione nel senso di Dickson si decompone in una somma diretta di sottomoduli ciclici. Si ottiene che un modulo di torsione secondo Dickson su un *d*-anello è una somma diretta di sottomoduli ognuno dei quali è canonicamente un modulo di torsione (nel senso usuale) su un dominio di valutazione discreta completo o su un anello locale Artiniano a ideali principali. Si estendono poi alcuni risultati già noti per altre teorie della torsione alla teoria della torsione di Dickson sui *d*-anelli. In particolare, denotato con $R_{\mathfrak{D}}$ l'anello dei quozienti di R rispetto alla topologia di Dickson $\mathfrak{D}$, si studia sotto quali ipotesi il modulo $R_{\mathfrak{D}}/R$ abbia anello degli endomorfismi commutativo e quando tale anello degli endomorfismi coincida con il completamento di Hausdorff di R nella topologia $\mathfrak{D}$.

All rings we consider are commutative with identity and all modules are unitary. Recall that a module is said to be *torsion in the sense of Dickson* (or $\mathfrak{D}$-*torsion*) if every non-zero homomorphic image contains a simple submodule and that a ring is said to be a T-*ring* if every $\mathfrak{D}$-torsion module has a primary decomposition, i.e. if every $\mathfrak{D}$-torsion module is a direct sum of submodules each of which has every non-zero homomorphic image containing a simple module isomorphic to some fixed simple module (see [2]). Recall also that a ring R is called a *d-ring* ([11]) if every finitely generated $\mathfrak{D}$-torsion R-module is a direct sum of cyclic submodules.

d-rings have been studied by T. S. Shores ([11]; see also [12]); he has characterized *d*-rings as those T-rings R such that $\mathfrak{M}/\mathfrak{M}^2$ is a cyclic module for every maximal ideal $\mathfrak{M}$ in R. From this characterization and from [1], Satz 5, we immediately deduce that a Noetherian ring R is a *d*-ring if and only if R is a direct product of a finite number of Dedekind domains and local Artinian principal ideal rings. Therefore we have at our disposal a complete description of Noetherian *d*-rings. Now we want to study (non-Noetherian) *d*-rings and in particular $\mathfrak{D}$-torsion modules over *d*-rings. Let su begin with a description of the $\mathfrak{D}$-injective envelopes $E_{\mathfrak{D}}(S)$ of the simple modules S of a *d*-ring R (see [16], Ch. IX, § 2). The idea is in fact the following: in the study of $\mathfrak{D}$-torsion modules the modules $E_{\mathfrak{D}}(S)$ can play the role that the Prüfer

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groups $\mathbf{Z}_{p^\infty}$ have in the study of abelian groups; from the following proposition we have in fact that the structure of such modules is exactly that of the groups $\mathbf{Z}_{p^\infty}$.

PROPOSITION 1. (Structure of the $\mathfrak{D}$-injective envelopes of the simple modules). *Let* R *be a d-ring,* $\mathfrak{M}$ *a maximal ideal in* R, $E = E_{\mathfrak{D}}(R/\mathfrak{M})$ *the* $\mathfrak{D}$*-injective envelope of* $R/\mathfrak{M}$. *For every* $n \geq 0$ *let* $A_n = (0 :_E \mathfrak{M}^n)$. *Then*:

i) $E = \bigcup_{n \geq 0} A_n$;

ii) *every* A_n *is a cyclic module of finite length*;

iii) *every proper submodule of* E *is of the type* A_n *for some* $n \geq 0$;

iv) E *is a serial module* (see [14]).

If furthermore $\mathfrak{M}^n \neq \mathfrak{M}^{n-1}$, *then*:

v) $A_n \neq A_{n-1}$;

vi) A_n *is generated by every element in* $A_n \setminus A_{n-1}$;

vii) $A_n \cong R/\mathfrak{M}^n$.

Proof. i) Since $R/\mathfrak{M}$ is a $\mathfrak{D}$-torsion module, its $\mathfrak{D}$-injective envelope $E = E_{\mathfrak{D}}(R/\mathfrak{M})$ coincides with the $\mathfrak{D}$-torsion submodule of the injective envelope $E(R/\mathfrak{M})$ of $R/\mathfrak{M}$. Since R is a *d*-ring, $E = L_\omega(E(R/\mathfrak{M}))$ (see [11], Cor.). Therefore every element of E is annihilated by some power of $\mathfrak{M}$. From this i) follows.

v) Let us show that if $\mathfrak{M}^n \neq \mathfrak{M}^{n-1}$, then $A_n \neq A_{n-1}$. If $\mathfrak{M}^n \neq \mathfrak{M}^{n-1}$, $\mathfrak{M}^{n-1}/\mathfrak{M}^n$ is a non-zero semisimple module. Hence there exists a non-zero homomorphism $\varphi : \mathfrak{M}^{n-1}/\mathfrak{M}^n \to R/\mathfrak{M}$; φ extends to a homomorphism $\psi : R/\mathfrak{M}^n \to E_{\mathfrak{D}}(R/\mathfrak{M})$ because $R/\mathfrak{M}^n$ is a $\mathfrak{D}$-torsion module. Let $\pi : R \to R/\mathfrak{M}^n$ be the canonical projection; then $\ker(\psi\pi) \supseteq \mathfrak{M}^n$ and $\ker(\psi\pi) \not\supseteq \mathfrak{M}^{n-1}$. Let $x = \psi\pi(1) \in E_{\mathfrak{D}}(R/\mathfrak{M})$. Then $x \in A_n$ and $x \notin A_{n-1}$.

vi) Let us show that if $A_n \neq A_{n-1}$, then A_n is generated by every element in $A_n \setminus A_{n-1}$. Induction on n. For $n = 1$ the statement is trivial. Hence suppose $n > 1$, $A_n \neq A_{n-1}$ and that if $A_{n-1} \neq A_{n-2}$, then A_{n-1} is generated by every element in $A_{n-1} \setminus A_{n-2}$. From $A_n \neq A_{n-1}$, it follows that $\mathfrak{M}^n \neq \mathfrak{M}^{n-1}$ and hence a fortiori $\mathfrak{M}^{n-1} \neq \mathfrak{M}^{n-2}$. From v) it follows that $A_{n-1} \neq A_{n-2}$, and therefore A_{n-1} is generated by every element in $A_{n-1} \setminus A_{n-2}$. Let us show that the module A_n/A_{n-1} is simple. It is semisimple because it is annihilated by $\mathfrak{M}$. If it were not simple, it would contain two simple submodules B and C such that $B \cap C = 0$. Let $x, y \in A_n$ whose residue classes modulo A_{n-1} generate B and C respectively. Then $Rx + Ry$ is a finitely generated $\mathfrak{D}$-torsion module. Since R is a *d*-ring, $Rx + Ry$ is a direct sum of cyclic modules $C_\lambda \neq 0$. But the C_λ's are submodules of E which is an essential extension of $R/\mathfrak{M}$. Hence every C_λ contains $R/\mathfrak{M}$, and thus $Rx + Ry$ itself is a cyclic module. But then $B + C$ is cyclic. Since $B + C$ is semisimple, it is simple, and this is a contradiction because $B + C = B \oplus C$. Hence A_n/A_{n-1} is simple. Now let $z \in A_n \setminus A_{n-1}$ and let us prove that z generates A_n. We have that $\mathfrak{M}z \nsubseteq A_{n-2}$ (otherwise $z \in A_{n-1}$). Let $m \in \mathfrak{M}$ be such that $mz \notin A_{n-2}$. Since $mz \in A_{n-1}$, by what previously seen mz generates A_{n-1}. It follows that $Rz \supseteq A_{n-1}$. Since $z \notin A_{n-1}$, we have that $A_n \supseteq Rz \supsetneqq A_{n-1}$, and therefore $A_n = Rz$ because A_n/A_{n-1} is simple. This proves that A_n is generated by every element in $A_n \setminus A_{n-1}$.

vii) By vi) A_n is cyclic; hence it suffices to show that if $\mathfrak{M}^n \neq \mathfrak{M}^{n-1}$, then $\operatorname{Ann} A_n = \mathfrak{M}^n$. Induction on n. If $n = 1$, trivial. Therefore suppose $n > 1$ and $\mathfrak{M}^n \neq \mathfrak{M}^{n-1}$. Then $\mathfrak{M}^{n-1} \neq \mathfrak{M}^{n-2}$ and hence by the inductive hypothesis $\operatorname{Ann} A_{n-1} = \mathfrak{M}^{n-1}$. Therefore $\mathfrak{M}^n \subseteq \operatorname{Ann} A_n \subseteq \operatorname{Ann} A_{n-1} = \mathfrak{M}^{n-1}$. On the other hand $\operatorname{Ann} A_n \neq \mathfrak{M}^{n-1}$, otherwise $A_n = A_{n-1}$, contradiction by v). Hence $\mathfrak{M}^n \subseteq \operatorname{Ann} A_n \subsetneqq \mathfrak{M}^{n-1}$. Since R is a d-ring, by [1], Hilfssatz 1, we conclude that $\operatorname{Ann} A_n = \mathfrak{M}^n$.

ii) trivially follows from vii) and [1], Hilfssatz 1.

iii) Let M be a proper submodule of E. Then by i) $\bigcup_{n \geq 0} A_n \nsubseteq M$. Let n be the smallest number such that $A_n \nsubseteq M$. It is then easy to show that $M = A_{n-1}$ by vi).

iv) Obvious by iii). QED.

Recall that if S is a simple R-module, an R-module M is said S-*primary* if every non-zero homomorphic image of M contains a submodule isomorphic to S.

COROLLARY 1. *Let* R *be a d-ring,* $\mathfrak{M}$ *a maximal ideal in* R.

i) *If there exist positive integers n such that $\mathfrak{M}^n = \mathfrak{M}^{n+1}$, let $\bar{n}$ be the smallest such n. Then $E_{\mathfrak{D}}(R/\mathfrak{M}) \cong R/\mathfrak{M}^{\bar{n}}$ is a cyclic serial module of finite length $\bar{n}$. In this case every $R/\mathfrak{M}$-primary R-module is a direct sum of cyclic modules isomorphic to $R/\mathfrak{M}^k$ for some $k \leq \bar{n}$.*

ii) *If for every $n > 0$ $\mathfrak{M}^n \neq \mathfrak{M}^{n+1}$, then $E_{\mathfrak{D}}(R/\mathfrak{M})$ is a serial Artinian non-Noetherian module.*

Proof. i) In this case the A_n are all distinct for $n \leq \bar{n}$ and $E_{\mathfrak{D}}(R/\mathfrak{M}) = A_{\bar{n}}$. Hence $E_{\mathfrak{D}}(R/\mathfrak{M}) \cong R/\mathfrak{M}^{\bar{n}}$ is a cyclic serial module of length $\bar{n}$. If M is a $R/\mathfrak{M}$-primary module, every element of M is annihilated by a power of $\mathfrak{M}$, and hence by $\mathfrak{M}^{\bar{n}}$. It follows that M is a $R/\mathfrak{M}^{\bar{n}}$-module. But $R/\mathfrak{M}^{\bar{n}}$ is a local Artinian principal ideal ring and hence every $R/\mathfrak{M}^{\bar{n}}$-module is a direct sum of cyclic submodules ([10], Th. 6.7). Hence M is a direct sum of cyclic R-modules.

ii) Obvious. QED.

Now that we have a good description of the module $E_{\mathfrak{D}}(R/\mathfrak{M})$ (prop. 1), let us examine the structure of its endomorphism ring: such a ring is the analogue of the ring of p-adic integers for abelian p-groups. The closeness of such an analogy is put into light by the following proposition.

PROPOSITION 2. (Structure of the endomorphism ring of $E_{\mathfrak{D}}(R/\mathfrak{M})$). *Let* R *be a d-ring*, $\mathfrak{M}$ *a maximal ideal in* R. *Then* $\mathrm{End}_R(E_{\mathfrak{D}}(R/\mathfrak{M}))$ *is the Hausdorff completion of* R *in the* $\mathfrak{M}$*-adic topology. If* $\mathfrak{M}^n \neq \mathfrak{M}^{n+1}$ *for every* $n > 0$, *then* $\mathrm{End}_R(E_{\mathfrak{D}}(R/\mathfrak{M}))$ *is a complete discrete valuation domain.*

Proof.

$$\mathrm{End}_R(E_{\mathfrak{D}}(R/\mathfrak{M})) \cong \mathrm{Hom}_R(\varinjlim_n A_n, E_{\mathfrak{D}}(R/\mathfrak{M})) \cong \varprojlim_n \mathrm{Hom}_R(A_n, E_{\mathfrak{D}}(R/\mathfrak{M})) \cong$$

$$\cong \varprojlim_n \mathrm{Hom}_R(A_n, A_n) \cong \varprojlim_n R/\mathfrak{M}^n \cong (R, \mathfrak{M})^\wedge,$$ where $(R, \mathfrak{M})^\wedge$ denotes the Hausdorff completion of R in the $\mathfrak{M}$-adic topology.

For the rest of the proof let us suppose that $\mathfrak{M}^n \neq \mathfrak{M}^{n+1}$ for all $n > 0$. Then $E_{\mathfrak{D}}(R/\mathfrak{M})$ is not Noetherian (cor. 1), whilst every proper submodule of $E_{\mathfrak{D}}(R/\mathfrak{M})$ is Noetherian (prop. 1, ii) and iii)). From this it is easy to deduce that every non-zero endomorphism of $E_{\mathfrak{D}}(R/\mathfrak{M})$ is surjective. Let us show that $\mathrm{End}_R(E_{\mathfrak{D}}(R/\mathfrak{M}))$ (which is certainly a commutative local ring, since it is isomorphic to the completion of R in the $\mathfrak{M}$–adic topology, and which is integral since every non-zero endomorphism is surjective) is a principal ideal ring. Let I be a non-zero ideal in $\mathrm{End}_R(E_{\mathfrak{D}}(R/\mathfrak{M}))$ and let $n = \min\{k \mid$ there exists $f \in I$ such that $\ker f = A_k\}$. Let $g \in I$, $g \neq 0$, with $\ker g = A_n$. Let us show that $(g) = I$. Certainly $(g) \subseteq I$. Let $f \in I$, $f \neq 0$. Then $\ker f = A_m$ with $m \geq n$. Since f and g are non-zero, they are surjective and hence they induce two isomorphisms $\tilde{f}: E_{\mathfrak{D}}(R/\mathfrak{M})/A_m \to E_{\mathfrak{D}}(R/\mathfrak{M})$ and $\tilde{g}: E_{\mathfrak{D}}(R/\mathfrak{M})/A_n \to E_{\mathfrak{D}}(R/\mathfrak{M})$. Let $\pi: E_{\mathfrak{D}}(R/\mathfrak{M})/A_n \to E_{\mathfrak{D}}(R/\mathfrak{M})/A_m$ be the canonical projection. Then $h = \tilde{f}\pi\tilde{g}^{-1} \in \mathrm{End}_R(E_{\mathfrak{D}}(R/\mathfrak{M}))$ and $hg = f$. It follows that $f \in (g)$. Hence $(g) = I$ and $\mathrm{End}_R(E_{\mathfrak{D}}(R/\mathfrak{M}))$ is a local principal ideal domain. The verification that it is complete is routine.

QED.

Now let R be a d-ring and $\mathfrak{M}$ a maximal ideal in R such that $\mathfrak{M}^n \neq \mathfrak{M}^{n-1}$ for all n. Let $(R, \mathfrak{M})^\wedge$ be the Hausdorff completion of R in the $\mathfrak{M}$-adic topology. By what we have just seen, $(R, \mathfrak{M})^\wedge$ is a complete DVR. Let $\varphi: R \to (R, \mathfrak{M})^\wedge$ be the canonical homomorphism and let T be a torsion $(R, \mathfrak{M})^\wedge$-module (torsion in the usual sense). Via φ, T has an R-module structure. Since $(R, \mathfrak{M})^\wedge$ is a DVR, every element of T is annihilated by some power of the maximal ideal $\hat{\mathfrak{M}}$ of $(R, \mathfrak{M})^\wedge$, and hence by some power of the maximal ideal $\varphi^{-1}(\hat{\mathfrak{M}}) = \mathfrak{M}$ of R. Hence T is a $R/\mathfrak{M}$-primary R-module.

Conversely let T′ be a $R/\mathfrak{M}$-primary R-module and let $a \in (R, \mathfrak{M})^\wedge$, $x \in T'$. Then $\mathfrak{M}^n x = 0$ for some n and there exists $a' \in R$ such that $\varphi(a') - a \in \hat{\mathfrak{M}}^n$. If we define $ax = a'x$ it is easily verified that this definition makes T′ into a torsion $(R, \mathfrak{M})^\wedge$-module.

We have thus found an isomorphism between the full subcategory of R-Mod consisting of all $R/\mathfrak{M}$-primary R-modules and the full subcategory of $(R, \mathfrak{M})^\wedge$-Mod consisting of all torsion $(R, \mathfrak{M})^\wedge$-modules. This allows us to use the theory of torsion modules over complete DVR's in the study of $\mathfrak{D}$-torsion modules.

Let us collect these remarks, cor. 1, prop. 2 and [2], Th. 3.4, in the following theorem.

THEOREM. (Structure of $\mathfrak{D}$-torsion modules). *Let* R *be a d-ring*, Ω *the set of all maximal ideals of* R, M *an* R*-module. Then* M *is a* $\mathfrak{D}$*-torsion module if and only if* $M = \bigoplus_{\mathfrak{M} \in \Omega} M_{\mathfrak{M}}$ *and for every* $\mathfrak{M} \in \Omega$

i) *if* $\mathfrak{M}^n = \mathfrak{M}^{n+1}$ *for some* n, *then* $M_{\mathfrak{M}}$ *is isomorphic to a direct sum of cyclic modules of the type* $R/\mathfrak{M}^k$;

ii) *if* $\mathfrak{M}^n \neq \mathfrak{M}^{n+1}$ *for every* n *and* $(R, \mathfrak{M})^\wedge$ *is the Hausdorff completion of* R *in the* $\mathfrak{M}$*-adic topology, then* $M_{\mathfrak{M}}$ *is a torsion module over the complete* DVR $(R, \mathfrak{M})^\wedge$.

At this point it is easy to prove that the module $E_{\mathfrak{D}}(R/\mathfrak{M})$ from which we have begun our study, is nothing but the injective envelope of the module $R/\mathfrak{M} \cong (R, \mathfrak{M})^\wedge / \hat{\mathfrak{M}}$ as $(R, \mathfrak{M})^\wedge$-module.

APPLICATIONS

When $\mathfrak{F}$ is a Gabriel filter on a ring R, it is possible to define the ring of quotients $R_{\mathfrak{F}}$ of R with respect to $\mathfrak{F}$ (see e.g. [16] Ch. IX). If R has no $\mathfrak{F}$-torsion, the ring R embeds canonically into $R_{\mathfrak{F}}$ and it is therefore possible to consider the $\mathfrak{F}$-torsion module $K_{\mathfrak{F}} = R_{\mathfrak{F}}/R$. This module has been the object of much study, above all in the case in which the ring R satisfies Stenström's $\mathfrak{F}$-inv condition ([15]). In this case, for instance, it has been shown that the ring $\mathrm{End}_R(K_{\mathfrak{F}})$ is the Hausdorff completion of R in the topology $\mathfrak{F}$ ([15], Cor. 4.5), and the structure of $K_{\mathfrak{F}}$ has been studied in problems concerning duality ([9]).

Now let $\mathfrak{D}$ be the set of all ideals I in R such that R/I is a $\mathfrak{D}$-torsion R-module. $\mathfrak{D}$ is a Gabriel filter on R. Suppose R has no $\mathfrak{D}$-torsion and consider the $\mathfrak{D}$-torsion module $K_{\mathfrak{D}} = R_{\mathfrak{D}}/R$. In general R does not satisfy the $\mathfrak{D}$-inv condition, so that the results in [15] and [9] are not applicable. Nevertheless for d-rings it is possible to use our theorem in the study of the $\mathfrak{D}$-torsion module $K_{\mathfrak{D}} = R_{\mathfrak{D}}/R$.

Note the connections between the following proposition, [9] (§ 4, Th. 7.3 and 3.6) and [7] (Prop. 2.5).

PROPOSITION 3. *Let* R *be a d-ring without* $\mathfrak{D}$*-torsion*, $K_{\mathfrak{D}} = R_{\mathfrak{D}}/R$, Ω *the set of all maximal ideals of* R. *The following statements are equivalent*:

i) $\mathrm{End}_R(K_{\mathfrak{D}})$ *is a commutative ring*;

ii) $K_{\mathfrak{D}}$ *is isomorphic to a submodule of* $\bigoplus_{\mathfrak{M} \in \Omega} E_{\mathfrak{D}}(R/\mathfrak{M})$;

iii) *every submodule of* $K_{\mathfrak{D}}$ *is fully invariant*;

iv) *the* R*-module* $\mathrm{Hom}_R(\mathfrak{M}, R)$ *can be generated by two elements for every* $\mathfrak{M} \in \Omega$.

Proof. i) $\Rightarrow$ ii) $\mathrm{End}_R(K_{\mathfrak{D}}) \cong \mathrm{End}_R\left(\bigoplus_{\mathfrak{M}\in\Omega} (K_{\mathfrak{D}})_{\mathfrak{M}}\right) \cong \prod_{\mathfrak{M}\in\Omega} \mathrm{End}_R((K_{\mathfrak{D}})_{\mathfrak{M}})$, and hence every $(K_{\mathfrak{D}})_{\mathfrak{M}}$ is a $R/\mathfrak{M}$-primary module with commutative endomorphism ring. By the theorem every $(K_{\mathfrak{D}})_{\mathfrak{M}}$ is isomorphic to a direct sum of cyclic modules of type $R/\mathfrak{M}^n$ or is a torsion module on a DVR. Since $\mathrm{End}_R((K_{\mathfrak{D}})_{\mathfrak{M}})$ is commutative, from [6], Th. 9, it follows that $(K_{\mathfrak{D}})_{\mathfrak{M}}$ is isomorphic to $R/\mathfrak{M}^k$ for some k or to $E_{\mathfrak{D}}(R/\mathfrak{M})$. In both cases $(K_{\mathfrak{D}})_{\mathfrak{M}}$ is isomorphic to a submodule of $E_{\mathfrak{D}}(R/\mathfrak{M})$. ii) follows.

ii) $\Rightarrow$ iii) If $K_{\mathfrak{D}}$ is isomorphic to a submodule of $\bigoplus_{\mathfrak{M}\in\Omega} E_{\mathfrak{D}}(R/\mathfrak{M})$, every endomorphism of $K_{\mathfrak{D}}$ extends to a endomorphism of $\bigoplus_{\mathfrak{M}\in\Omega} E_{\mathfrak{D}}(R/\mathfrak{M})$. On the other hand every submodule of $\bigoplus_{\mathfrak{M}\in\Omega} E_{\mathfrak{D}}(R/\mathfrak{M})$ is fully invariant. iii) follows.

iii) $\Rightarrow$ i) Obvious.

ii) $\Rightarrow$ iv) Since $R_{\mathfrak{D}} = \varinjlim_{I\in\mathfrak{D}} \mathrm{Hom}_R(I, R)$, where the connecting homomorphisms are injective because R has no $\mathfrak{D}$-torsion, we deduce that $\mathrm{Hom}_R(I, R)$ is isomorphic to a submodule of $R_{\mathfrak{D}}$ for every $I \in \mathfrak{D}$. In particular $\mathrm{Hom}_R(\mathfrak{M}, R)$ is isomorphic to a submodule of $R_{\mathfrak{D}}$ for every $\mathfrak{M} \in \Omega$. Hence $\mathrm{Hom}_R(\mathfrak{M}, R)/R$ is isomorphic to a submodule of $K_{\mathfrak{D}}$ and hence of $\bigoplus_{\mathfrak{M}\in\Omega} E_{\mathfrak{D}}(R/\mathfrak{M})$ by ii). But $\mathrm{Hom}_R(\mathfrak{M}, R)/R$ is annihilated by $\mathfrak{M}$, and hence it is isomorphic to a submodule of the $\mathfrak{M}$-socle of $\bigoplus_{\mathfrak{M}\in\Omega} E_{\mathfrak{D}}(R/\mathfrak{M})$. But the $\mathfrak{M}$-socle of $\bigoplus_{\mathfrak{M}\in\Omega} E_{\mathfrak{D}}(R/\mathfrak{M})$ is simple, so $\mathrm{Hom}_R(\mathfrak{M}, R)/R$ is trivial or simple. In both cases $\mathrm{Hom}_R(\mathfrak{M}, R)$ can be generated by two elements.

iv) $\Rightarrow$ ii) Arguing as in [7], proof of Lemma 2.3, it is not difficult to show that if $\mathrm{Hom}_R(\mathfrak{M}, R)$ can be generated by two elements, then $\mathrm{Hom}_R(\mathfrak{M}, R)/R$ is trivial or simple. We have already seen in the proof of ii) $\Rightarrow$ iv) that $\mathrm{Hom}_R(\mathfrak{M}, R)/R$ is canonically isomorphic to a submodule of $K_{\mathfrak{D}}$. It is easily proved that the image of $\mathrm{Hom}_R(\mathfrak{M}, R)/R$ into $K_{\mathfrak{D}}$ is exactly the $\mathfrak{M}$-socle of $K_{\mathfrak{D}}$. Therefore the $\mathfrak{M}$-socle of $K_{\mathfrak{D}}$ is zero or simple. By the theorem and by [6], Th. 9, it follows that the $R/\mathfrak{M}$-primary submodule $(K_{\mathfrak{D}})_{\mathfrak{M}}$ of $K_{\mathfrak{D}}$ is isomorphic to a submodule of $E_{\mathfrak{D}}(R/\mathfrak{M})$. Hence $K_{\mathfrak{D}}$ is isomorphic to a submodule of $\bigoplus_{\mathfrak{M}\in\Omega} E_{\mathfrak{D}}(R/\mathfrak{M})$. QED.

It is now easy to prove the analogue of [15], Cor. 4.5, for the filter $\mathfrak{D}$.

COROLLARY 2. *Let* R *be a d-ring without* $\mathfrak{D}$*-torsion,* $K_{\mathfrak{D}} = R_{\mathfrak{D}}/R$. *Then* $\mathrm{End}_R(K_{\mathfrak{D}})$ *is the Hausdorff completion of* R *in the topology* $\mathfrak{D}$ *if and only if* $K_{\mathfrak{D}} \cong \bigoplus_{\mathfrak{M}\in\Omega} E_{\mathfrak{D}}(R/\mathfrak{M})$.

Proof. If $\mathrm{End}_R(K_{\mathfrak{D}})$ is the completion of R in the topology $\mathfrak{D}$, then by prop. 3, i) $\Rightarrow$ ii), $K_{\mathfrak{D}}$ is isomorphic to a submodule of $\bigoplus_{\mathfrak{M}\in\Omega} E_{\mathfrak{D}}(R/\mathfrak{M})$. Since the $K_{\mathfrak{D}}$-topology on R coincides with the topology $\mathfrak{D}$, for every $\mathfrak{M}\in\Omega$ and for every $n \geq 0$ there exists a finitely generated submodule L of $K_{\mathfrak{D}}$ such that $\mathfrak{M}^n \supseteq \mathrm{Ann}\, L$. By prop. 1 we conclude that $K_{\mathfrak{D}} \cong \bigoplus_{\mathfrak{M}\in\Omega} E_{\mathfrak{D}}(R/\mathfrak{M})$.

Conversely if $K_{\mathfrak{D}} \cong \bigoplus_{\mathfrak{M}\in\Omega} E_{\mathfrak{D}}(R/\mathfrak{M})$, by prop. 3 $\mathrm{End}_R(K_{\mathfrak{D}})$ is commutative. Since $K_{\mathfrak{D}} \cong \bigoplus_{\mathfrak{M}\in\Omega} E_{\mathfrak{D}}(R/\mathfrak{M})$ is a s.q.i. module (see [9], def. 3.1), $\mathrm{End}_R(K_{\mathfrak{D}})$ is the Hausdorff completion of R in the $K_{\mathfrak{D}}$-topology by [9], Th. 3.6. Since $K_{\mathfrak{D}} \cong \bigoplus_{\mathfrak{M}\in\Omega} E_{\mathfrak{D}}(R/\mathfrak{M})$, by prop. 1 the $K_{\mathfrak{D}}$-topology has the set of all products of a finite number of maximal ideals of R as a basis of neighbourhoods of 0. If we show that this set is a basis for the filter $\mathfrak{D}$, the corollary will be proved.

LEMMA. *Let* R *be a d-ring. Then the set of all products of a finite number of maximal ideals of* R *is a basis for the filter* $\mathfrak{D}$.

Proof. Let $\mathfrak{P}$ be the set in question. Certainly $\mathfrak{P} \subseteq \mathfrak{D}$. For the converse we have to prove that if $I \in \mathfrak{D}$, then I contains an element of $\mathfrak{P}$. By reducing modulo I, it is enough to show that in a semiartinian *d*-ring R the zero ideal is a product of maximal ideals. Now if R is a semiartinian *d*-ring, R is Artinian by [11], Corollary. Hence R has only a finite number of maximal ideals and their product is the Jacobson radical of R which is nilpotent. QED.

The next corollary (for the terminology see [9]) follows from cor. 2, [9] (Th. 7.3) and [15] (Cor. 4.5).

COROLLARY 3. *Let* R *be a d-ring without* $\mathfrak{D}$*-torsion. If* $\mathfrak{D}$ *is a Gabriel–Stenström filter, then* R *is a* $\mathfrak{D}$*-reflexive ring.*

EXAMPLES

i) A Noetherian ring R is a *d*-ring without $\mathfrak{D}$-torsion if and only if R is a direct product of a finite number of Dedekind domains which are not fields. In this case $R_{\mathfrak{D}}$ is the total ring of fractions of R (see [4], Th. 2.1) and $\mathfrak{D}$ is a Gabriel–Stenström filter. It follows that R is a $\mathfrak{D}$-reflexive ring, $R_{\mathfrak{D}}/R \cong \bigoplus_{\mathfrak{M}\in\Omega} E(R/\mathfrak{M})$, $\mathrm{End}_R(R_{\mathfrak{D}}/R)$ is the Hausdorff completion of R in the topology $\mathfrak{D}$ (which coincides with the R-topology, see [8], Ch. II), etc.

ii) Let C (X) be the ring of all continuous real valued functions on a completely regular topological space X. C (X) is a T-ring ([13], Th. 3) and $\mathfrak{M} = \mathfrak{M}^2$ for every maximal ideal $\mathfrak{M}$ of C (X). Hence C (X) is a *d*-ring and the $\mathfrak{D}$-torsion modules are exactly the semisimple modules. Furthermore it is easy to prove that C (X) has no $\mathfrak{D}$-torsion if and only if X has no isolated points. Let us show that $R = C(X)$ does not satisfy the equivalent conditions of prop. 3. In fact in this case $K_{\mathfrak{D}} = R_{\mathfrak{D}}/R$ is semisimple and hence is the direct sum of its $\mathfrak{M}$-socles, where $\mathfrak{M} \in \Omega$. As we have already seen in the proof of Prop. 3, iv) $\Rightarrow$ ii), the $\mathfrak{M}$-socle of $K_{\mathfrak{D}}$ is canonically isomorphic to $\mathrm{Hom}_R(\mathfrak{M}, R)/R$. Hence $K_{\mathfrak{D}} \cong \bigoplus_{\mathfrak{M} \in \Omega} \mathrm{Hom}_R(\mathfrak{M}, R)/R$. Now if $\mathfrak{M}$ is a free maximal ideal, then $\mathrm{Hom}_R(\mathfrak{M}, R) = R$. Therefore $K_{\mathfrak{D}} \cong \bigoplus_{\mathfrak{M} \in \Omega'} \mathrm{Hom}_R(\mathfrak{M}, R)/R$ where Ω' is the set of all fixed maximal ideals of $R = C(X)$. Now in general $\mathrm{Hom}_R(\mathfrak{M}, R)$ cannot be generated by two elements for all $\mathfrak{M} \in \Omega'$. For example if X satisfies the first axiom of countability and $\mathfrak{M}$ is the maximal ideal of $C(X)$ corrisponding to the point $p \in X$, $\mathrm{Hom}_R(\mathfrak{M}, R)$ is canonically isomorphic to the $C(X)$-module of all continuous functions $f: X \setminus \{p\} \to \mathbf{R}$ bounded in a neighbourhood of p, and it easy to show that such a module cannot be generated by two elements.

Remark. It is possible to prove the following proposition: "Let X be a completely regular topological space without isolated points, $C(X)$ the ring of all continuous real valued functions on X, $W(X)$ the ring of all functions $f: X \to \mathbf{R}$ continuous on all but a finite number of points $p_1, \cdots, p_n$ of X and with the property that $fg \in C(X)$ for every function $g \in C(X)$ such that $g(p_i) = 0$ for every $i = 1, \cdots, n$. Let $I(X)$ be the ideal in $W(X)$ consisting of all functions $f: X \to \mathbf{R}$ which are zero on all but a finite number of points of X. Then the ring of quotients $C(X)_{\mathfrak{D}}$ is canonically isomorphic to $W(X)/I(X)$". This is a topological description of the ring $C(X)_{\mathfrak{D}}$. It would be interesting to study the relations between the properties of X and of $C(X)_{\mathfrak{D}}$ in the style of what has been done for other rings of quotients in [5]. For instance it is possible to prove that: 1) $C(X) = C(X)_{\mathfrak{D}}$ if and only if every cofinite subspace of X is C^*-embedded in X; 2) $C(X)_{\mathfrak{D}}$ has no non-trivial idempotents if and only if every cofinite subspace of X is connected; 3) $C(\mathbf{R}^n)_{\mathfrak{D}} \not\cong C(\mathbf{R})_{\mathfrak{D}}$ for every $n > 1$; 4) Every prime ideal in $C(X)_{\mathfrak{D}}$ is absolutely convex; etc.

iii) Let R be a non-discrete archimedean valuation domain (i.e. a valuation domain with valuation group isomorphic to a dense subgroup of $\mathbf{R}$). Then R is a d-ring without $\mathfrak{D}$-torsion and $R_{\mathfrak{D}} = R$. Hence $K_{\mathfrak{D}} = 0$ and R trivially satisfies the equivalent conditions of Prop. 3, but not the ones of cor. 2.

iv) Consider the ring R of the example in [3], § 6, where $A = \mathbf{Z}$ is the ring of integers and $M = \mathbf{Q}$ is the group of rationals. Since Spec (R) and Spec ($\mathbf{Z}$) are homeomorphic, R is a T-ring by [13], Th. 1. The maximal ideals of R are principal generated by a regular element. Hence R is a non-Noetherian d-ring without $\mathfrak{D}$-torsion and the filter $\mathfrak{D}$ is Gabriel–Stenström. Therefore R satisfies the hypotheses of cor. 3.

REFERENCES

[1] K. ASANO (1951) – *Über kommutative Ringe in denen jedes Ideal als Produkt von Primeidealen darstellbar ist*, « J. Math. Soc. Japan », *3*, 82–90.

[2] S. E. DICKSON (1968) – *Decomposition of modules*. II. *Rings without chain conditions*, « Math. Z. », *104*, 349–357.

[3] A. FACCHINI (1979) – *Reflexive rings*, « Symposia Math. », XXIII, 415–449.

[4] A. FACCHINI (1980) – *On the ring of quotients of a Noetherian commutative ring with respect to the Dickson topology*, « Rend. Sem. Mat. Univ. Padova », *62*.

[5] N. J. FINE, L. GILLMAN and J. LAMBEK (1965) – *Rings of quotients of rings of functions*, McGill Univ. Press.

[6] I. KAPLANSKY (1952) – *Modules over Dedekind rings and valuation rings*, « Trans. Am. Math. Soc. », *72*, 327–340.

[7] E. MATLIS (1968) – *Reflexive domains*, « J. Algebra », *8*, 1–33.

[8] E. MATLIS (1970) – 1-*Dimensional Cohen–Macaulay Rings*, « Lecture Notes in Math. », *327*, Springer-Verlag Berlin-Heidelberg-New York.

[9] C. MENINI, and A. ORSATTI (1979) – *Duality over a quasi-injective module and commutative $\mathfrak{F}$-reflexive rings*, « Symposia Math. », XXIII, 145–179.

[10] D. W. SHARPE and P. VAMOS (1972) – *Injective modules*, Cambridge Univ. Press.

[11] T. S. SHORES (1971) – *Decompositions of finitely generated modules*, « Proc. Am. Math. Soc. », *30*, 445–450.

[12] T. S. SHORES (1972) – *The structure of Loewy modules*, « J. reine und angew. Math. », *254*, 204–220.

[13] T. S. SHORES (1976) – *A topological criterion for primary decomposition*, « Acta Math. Ac. Sc. Hungaricae », *28*, 383–387.

[14] T. S. SHORES and W. J. LEWIS (1974) – *Serial modules and endomorphism rings*, « Duke Math. J. », *41*, 889–909.

[15] B. STENSTRÖM (1970) – *On the completion of modules in an additive topology*, « J. Algebra », *16*, 523–540.

[16] B. STENSTRÖM (1975) – *Rings of Quotients*, Springer-Verlag.