Lawrence A. Harris, Jean-Pierre Vigué

A metric condition for equivalence of domains

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **67** (1979), n.6, p. 402–403.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1979_8_67_6_402_0>

Analisi funzionale. — *A metric condition for equivalence of domains.* Nota di LAWRENCE A. HARRIS e JEAN-PIERRE VIGUÉ, presentata (*) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si ottengono condizioni elementari per la bi-olomorficità di un'applicazione olomorfa da un dominio in uno spazio di Banach a un altro.

PROPOSITION 1. *Let $\mathscr{D}_1$ and $\mathscr{D}_2$ be domains in real Banach spaces and let $f : \mathscr{D}_1 \to \mathscr{D}_2$ be a continuously differentiable function such that* $\mathrm{D}f(x)^{-1}$ *exists for some* $x \in \mathscr{D}_1$. *Suppose* ρ_1 *and* ρ_2 *are metrics on* $\mathscr{D}_1$ *and* $\mathscr{D}_2$, *respectively, satisfying*

$$\rho_2(f(x), f(y)) \geq \rho_1(x, y), \qquad x, y \in \mathscr{D}_1 .$$

Suppose also that $\mathscr{D}_1$ *is (Cauchy) complete in* ρ_1, *that the norm is* ρ_1-*continuous on* $\mathscr{D}_1$ *and that for each* $x_0 \in \mathscr{D}_1$ *and* $w_0 \in \mathscr{D}_2$, *there exist positive numbers* δ, m *and* M *such that*

$$m \| x - x_0 \| \leq \rho_1(x, x_0) \quad , \quad \rho_2(w, w_0) \leq \mathrm{M} \| w - w_0 \|$$

whenever $\| x - x_0 \| < \delta$ *and* $\| w - w_0 \| < \delta$. *Then* f *is a diffeomorphism of* $\mathscr{D}_1$ *onto* $\mathscr{D}_2$.

It follows from [3, p. 368] and [1; Th. 8, Th. 24, Th. 19, (13)] that the hypotheses on $\mathscr{D}_1$, $\mathscr{D}_2$, ρ_1 and ρ_2 are satisfied when

I) $\mathscr{D}_1$ and $\mathscr{D}_2$ are bounded domains in $\mathbf{C}^n$, where $\mathscr{D}_1$ is a generalized analytic polyhedron or a strongly pseudoconvex domain with C^2 boundary, and ρ_1 and ρ_2 are the Bergman, Caratheodory, CRF or Kobayashi pseudometrics for $\mathscr{D}_1$ and $\mathscr{D}_2$, respectively,

and when

II) $\mathscr{D}_1$ and $\mathscr{D}_2$ are domains in a complex Banach space, where $\mathscr{D}_1$ is a bounded homogeneous domain or a convex domain with a finite interior point, and ρ_1 and ρ_2 are the Caratheodory, CRF or Kobayashi pseudometrics for $\mathscr{D}_1$ and $\mathscr{D}_2$, respectively.

(*) Nella seduta del 15 dicembre 1979.

Thus we have the following:

COROLLARY 2. *Suppose* (I) *or* (II) *holds and let* $f: \mathscr{D}_1 \to \mathscr{D}_2$ *be a holomorphic function such that* $Df(x)^{-1}$ *exists for some* $x \in \mathscr{D}_1$. *If*

$$\rho_2(f(x), f(y)) \geq \varepsilon\rho_1(x, y), \qquad x, y \in \mathscr{D}_1$$

for some $\varepsilon > 0$, *then* f *is a biholomorphic mapping of* $\mathscr{D}_1$ *onto* $\mathscr{D}_2$.

Proof. Let $A = \{x \in \mathscr{D}_1 : Df(x)^{-1} \text{ exists}\}$ and note that A is a non-empty open subset of $\mathscr{D}_1$. To see that A is closed in $\mathscr{D}_1$, let $a \in \bar{A} \cap \mathscr{D}_1$ and suppose $a \notin A$. Then by [2, Th. 4.11.1] there exists a sequence $\{v_n\}$ of unit vectors with $Df(a)v_n \to 0$. But by hypothesis,

$$M\|f(a+tv) - f(a)\| \geq \rho_2(f(a+tv), f(a)) \geq m\,|t|$$

for all unit vectors v and all small enough t, so $M\|Df(a)v\| \geq m\|v\|$, a contradiction. Thus $A = \mathscr{D}_1$ and hence f is a diffeomorphism of $\mathscr{D}_1$ onto $f(\mathscr{D})_1$ by the inverse function theorem.

Clearly $f(\mathscr{D}_1)$ is open in $\mathscr{D}_2$. To see that $f(\mathscr{D}_1)$ is closed in $\mathscr{D}_2$, let $b \in \overline{f(\mathscr{D}_1)} \cap \mathscr{D}_2$ and let $\{b_n\}$ be a sequence in $f(\mathscr{D}_1)$ which converges to b. Then $\{b_n\}$ is a ρ_2-Cauchy sequence in $\mathscr{D}_2$ and hence $\{c_n\}$ is a ρ_1-Cauchy sequence in $\mathscr{D}_1$, where $c_n = f^{-1}(b_n)$. Let c be the ρ_1-limit of this sequence. Then $\{c_n\}$ converges to c so $f(c) = b$ by the continuity of f. Thus $f(\mathscr{D}_1) = \mathscr{D}_2$.

Note that case (II) of Corollary 2 gives an affirmative answer for Banach spaces to a question raised in [1, Prob. 8, p. 403]. When $\mathscr{D}_1$ and $\mathscr{D}_2$ are domains in n-dimensional space, the assumption that $Df(x)^{-1}$ exists can be omitted from the statements of our results by the last inequality.

REFERENCES

[1] L. A. HARRIS (1979) - *Schwarz-Pick systems of pseudometrics for domains in normed linear spaces*, Advances in Holomorphy, J. A. Barroso editor, North-Holland, Amsterdam 305-406.

[2] E. HILLE and R. S. PHILLIPS (1957) - *Functional Analysis and Semi-groups*, «Amer. Math. Soc. Colloq. Publ.», *31*, Providence.

[3] S. KOBAYASHI (1976) - *Intrinsic distances, measures and geometric function theory*, «Bull. Amer. Math. Soc.», *82*, 357-416.