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## Certain dual series equations involving Jacobi polynomials

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**Funzioni speciali.** — *Certain dual series equations involving Jacobi polynomials* (*). Nota I di HARI M. SRIVASTAVA, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore dimostra che alcuni risultati ottenuti precedentemente da N. K. Thakare [*Zeitschr. Angew. Math. Mech.*, 54 (1974), 283–284] seguono facilmente da altri noti risultati.

L'Autore studia poi una generale classe di coppie di serie duali collegate ai polinomi di Jacobi.

## 1. INTRODUCTION

Various pairs of dual equations involving, for instance, trigonometric series, the Fourier-Bessel series, the Fourier-Legendre series, the Dini series, and series of Jacobi and Laguerre polynomials, arise in the investigation of certain classes of mixed boundary value problems in potential theory (cf. [6], Chapter V; *see also* [8]); for example, when one makes use of an appropriate orthogonal series for the axisymmetric solution $V(\rho, z)$ of Laplace's equation in the semi-infinite cylinder $0 \leqq \rho \leqq a$, $z \geqq 0$ satisfying the following boundary conditions:

$$(i)\quad V(\rho, z) \rightarrow 0 \quad \text{as} \quad z \rightarrow \infty,$$

$$(ii)\quad V(a, z) = 0, \qquad 0 \leqq z < \infty,$$

$$(iii)\quad V(\rho, 0) = f(\rho), \qquad 0 \leqq \rho < 1,$$

and

$$(iv)\quad \{\partial V/\partial z\}|_{z=0} = 0, \qquad 1 < \rho \leqq a.$$

Dual series equations in which the kernels involve Jacobi polynomials of the *same* indices were first considered by Noble [4], who used Jacobi's original notation:

$$\mathfrak{F}_n(a, \lambda; \rho) = {}_2F_1[-n, a+n; \lambda; \rho], \tag{1}$$

which can be found, among other places, in the early editions of the book by Magnus and Oberhettinger [3]. On the other hand, Srivastav [7] used

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Szegö's notation (cf. [9], p. 62):

$$(2) \qquad P_n^{(\alpha,\beta)}(x) = \binom{\alpha+n}{n} {}_2F_1\left[-n\,,\alpha+\beta+n+1\,;\alpha+1\,;\frac{1-x}{2}\right]$$

in order to solve a *special* case of Noble's equations. An account of both Noble's and Srivastav's solutions can be found in the recent book by Sneddon ([6], pp. 165–172), where the aforementioned connection has not been stated explicitly. {Indeed, from (1) and (2) it follows at once that

$$(3) \qquad P_n^{(\alpha,\beta)}(1-2x) = \binom{\alpha+n}{n} \mathfrak{F}_n(\alpha+\beta+1\,,\alpha+1\,;x)\,,$$

which will evidently lead to the fact that Srivastav's equations are essentially a special case of Noble's equations (with, of course, $a=\alpha+\beta+1$ and $\lambda=\alpha+1$) when $\mu=\beta+1/2$.}

A generalization of Noble's equations (cf. [4], p. 363) was considered subsequently by Dwivedi [1], whose equations involve Jacobi polynomials of different indices in the original notation (1). By elementary changes of variables and parameters, using the relationship (3), Dwidedi's equations (1.1) and (1.2) in reference [1, p. 287] (with $a=\alpha+\beta+1$, $\lambda=\alpha+1$, $r=\delta+1$, and $\mu$ replaced by $\mu+1$) can easily be put in their *equivalent forms*:

$$(4) \qquad \sum_{n=0}^{\infty} A_n^* \frac{\Gamma(\mu+n+1)}{\Gamma(\beta+n+1)} P_n^{(\alpha,\beta)}(x) = f^*(x)\,, \qquad -1 \leqq x < y\,,$$

$$(5) \qquad \sum_{n=0}^{\infty} A_n^* \frac{\Gamma(\alpha+\beta-\mu+n+1)}{\Gamma(\alpha+\beta-\delta+n+1)} P_n^{(\alpha+\beta-\delta,\delta)}(x) = g^*(x)\,, \quad y < x \leqq 1\,,$$

which would readily correspond to Noble's equations [4, p. 363] when $\delta=\beta$. Indeed, Dwivedi [1] determined the unknown sequence $\{A_n^*\}$ in terms of the prescribed functions $f^*(x)$ and $g^*(x)$ under the following alternative sets of conditions:

$$(6) \qquad \begin{cases} (i) & \alpha+\beta+1 > \delta > \alpha > \mu > -1\,; \qquad \text{or} \\ (ii) & \alpha+\beta+1 > \delta > \mu > \alpha-1 > -2\,. \end{cases}$$

{Here $A_n^*$, $f^*$ and $g^*$ are suitably related to Dwivedi's $A_n$, $f$ and $g$, respectively.}

Recently, Thakare [10] solved certain dual Jacobi series equations [using Szegö's notation (2)], which can at once be rewritten in their *equivalent* forms:

$$(7) \qquad \sum_{n=0}^{\infty} A_n^* \frac{\Gamma(\beta+\mu+n+1)}{\Gamma(\beta+n+1)} P_n^{(\alpha,\beta)}(x) = f^*(x)\,, \qquad -1 \leqq x < y\,,$$

$$(8)\qquad \sum_{n=0}^{\infty} A_n^* \frac{\Gamma(\alpha-\mu+n+1)}{\Gamma(\alpha+\beta-\delta+n+1)} P_n^{(\alpha+\beta-\delta,\delta)}(x) = g^*(x)\,, \qquad y < x \leqq 1\,,$$

where, in terms of the symbols used in Thakare's paper [10, p. 283, Eqs. (1) and (2)],

$$(9)\qquad \begin{cases} A_n^* = A_n/\{\Gamma(\alpha-\mu+n+1)\,\Gamma(\beta+\mu+n+1)\}\,, \\ f^*(x) = f(x)/\Gamma(\beta+1)\,, \qquad \text{and} \quad g^*(x) = g(x)/\Gamma(\alpha+\beta-\delta+1). \end{cases}$$

Evidently, these last equations (7) and (8) with the parameter $\mu$ replaced *trivially* by $\mu-\beta$ are the same as the dual series equations (4) and (5), whose exact solution was given earlier by Dwivedi [1] under two alternative sets of conditions in (6) above. Thus it is easy to derive Thakare's solution (cf. [10], p. 284) from either one of Dwivedi's solutions (cf. [1], p. 289, Eq (2.6); *see also* § 3) by merely making the aforeindicated trivial changes of variables and parameters and using the relationship (3). We omit the details involved, but remark in passing that Thakare's solvability conditions $\mu > 0$ and $\delta-\mu-\beta > 0$ are mutually contradictory in the special case $\delta = \beta$, considered by Thakare [10, p. 284, § 5], which would obviously reduce (7) and (8) to Noble's equations (and hence also to Srivastav's equations).

## 2. Generalization of the dual Equations (7) and (8)

By applying an interesting modification of the familiar multiplying-factor technique, developed by Noble [4], we shall consider the problem of determining the unknown sequence $\{A_n\}$ satisfying the general dual series equations:

$$(10)\qquad \sum_{n=0}^{\infty} A_n \frac{\Gamma(\mu+n+l+1)}{\Gamma(\beta+n+l+1)} P_{n+l}^{(\alpha,\beta)}(x) = f(x)\,, \qquad -1 \leqq x < y\,,$$

$$(11)\qquad \sum_{n=0}^{\infty} A_n \frac{\Gamma(\lambda+n+l+1)}{\Gamma(\gamma+n+l+1)} P_{n+l}^{(\gamma,\delta)}(x) = g(x)\,, \qquad y < x \leqq 1\,,$$

where $l$ is an arbitrary non-negative integer, $f(x)$ and $g(x)$ are prescribed functions, and, in general,

$$(12)\qquad \min\{\alpha, \beta, \gamma, \delta, \lambda, \mu\} > -1\,.$$

The following results involving Jacobi polynomials will be required in the course of our investigation.

(*i*) The orthogonality property of the Jacobi polynomials given by [cf., e.g., [9], p. 68, Eq. (4.3.3)]:

$$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) \, dx \tag{13}$$

$$= \frac{2^{\alpha+\beta+1} \Gamma(\alpha+n+1) \Gamma(\beta+n+1)}{n! (\alpha+\beta+2n+1) \Gamma(\alpha+\beta+n+1)} \delta_{mn}, \qquad \alpha > -1 \;,\quad \beta > -1,$$

where $\delta_{mn}$ is the Kronecker delta.

(*ii*) The following convenient forms of the known fractional integrals (44) and (43) on page 191 in reference [2]:

$$\int_{-1}^{\xi} (1+x)^{\beta} (\xi - x)^{\rho-1} P_n^{(\alpha,\beta)}(x) \, dx \tag{14}$$

$$= B(\beta+n+1, \rho) (1+\xi)^{\beta+\rho} P_n^{(\alpha-\rho,\beta+\rho)}(\xi), \qquad \beta > -1 \;,\quad \rho > 0,$$

and

$$\int_{\xi}^{1} (1-x)^{\alpha} (x-\xi)^{\sigma-1} P_n^{(\alpha,\beta)}(x) \, dx \tag{15}$$

$$= B(\alpha+n+1, \sigma) (1-\xi)^{\alpha+\sigma} P_n^{(\alpha+\sigma,\beta-\sigma)}(\xi), \qquad \alpha > -1 \;,\quad \sigma > 0,$$

where $B(\alpha, \beta)$ denotes the familiar Beta function defined by

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad \alpha > 0, \beta > 0. \tag{16}$$

(*iii*) For integers $m \geqq 0$, we have the derivative formula

$$D_x^m \{(1-x)^{\alpha+m} P_n^{(\alpha+m,\beta-m)} x)\} \tag{17}$$

$$= (-1)^m \frac{\Gamma(\alpha+m+n+1)}{\Gamma(\alpha+n+1)} (1-x)^{\alpha} P_n^{(\alpha,\beta)}(x), \qquad D_x = d/dx,$$

which follows from the known results (7), p. 264 and (17), p. 265 in reference [5], and its complement

$$D_x^m \{(1+x)^{\beta+m} P_n^{(\alpha-m,\beta+m)}(x)\} \tag{18}$$

$$= \frac{\Gamma(\beta+m+n+1)}{\Gamma(\beta+n+1)} (1+x)^{\beta} P_n^{(\alpha,\beta)}(x),$$

which can be proven by using the known formulas (5), p. 264 and (17), p. 265 in reference [5].

{Incidentally, since it is fairly well known that [9, p. 59, Eq. (4.1.3)]

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x), \tag{19}$$

the integral formulas (14) and (15) are essentially equivalent, and so are the derivative formulas (17) and (18)}.

Now we turn to our series equation (10), multiply it by $(1+x)^{\beta}(\xi-x)^{\rho+m-1}$, whre $\rho$ is a suitable constant and $m$ is a non-negative integer, and then integrate both sides with respect to $x$ over the interval $(-1, \xi)$. Making use of the integral formula (14), we thus have

$$\sum_{n=0}^{\infty} A_n \frac{\Gamma(\mu+n+l+1)}{\Gamma(\beta+\rho+m+n+l+1)} P_{n+l}^{(\alpha-\rho-m,\beta+\rho+m)}(\xi) = \frac{(1+\xi)^{-\beta-\rho-m}}{\Gamma(\rho+m)} \int_{-1}^{\xi} (1+x)^{\beta}(\xi-x)^{\rho+m-1} f(x)\, dx, \tag{20}$$

where $-1<\xi<y$, $\beta>-1$, and $\rho+m>0$.

On multiplying this last equation (20) by $(1+\xi)^{\beta+\rho+m}$, if we differentiate both sides $m$ times with respect to $\xi$ using the derivative formula (18), we shall get

$$\sum_{n=0}^{\infty} A_n \frac{\Gamma(\mu+n+l+1)}{\Gamma(\beta+\rho+n+l+1)} P_{n+l}^{(\alpha-\rho,\beta+\rho)}(\xi) = \frac{(1+\xi)^{-\beta-\rho}}{\Gamma(\rho+m)} D_{\xi}^{m} \left\{ \int_{-1}^{\xi} (1+x)^{\beta}(\xi-x)^{\rho+m-1} f(x)\, dx \right\}, \tag{21}$$

where, as before, $-1<\xi<y$, $\beta>-1$, and $\rho+m>0$, $m$ being a non-negative integer.

Next we multiply our equation (11) by $(1-x)^{\gamma}$ and differentiate both sides $k$ times with respect to $x$, using the derivative formula (17); we thus obtain

$$\sum_{n=0}^{\infty} A_n \frac{\Gamma(\lambda+n+l+1)}{\Gamma(\gamma-k+n+l+1)} P_{n+l}^{(\gamma-k,\delta+k)}(x) = (-1)^{k}(1-x)^{k-\gamma} D_x^{k}\{(1-x)^{\gamma} g(x)\}, \qquad y<x\leqq 1, \tag{22}$$

where $k$ is a non-negative integer.

If we multiply the preceding equation (22) by $(1-x)^{\gamma-k}(x-\xi)^{\sigma+k-1}$, where $\sigma$ is a suitable constant, and then integrate both sides with respect to $x$ over the interval $(\xi, 1)$ using (15), we find that

$$\sum_{n=0}^{\infty} A_n \frac{\Gamma(\lambda+n+l+1)}{\Gamma(\gamma+\sigma+n+l+1)} P_{n+l}^{(\gamma+\sigma,\delta-\sigma)}(\xi) = \frac{(-1)^{k}(1-\xi)^{-\gamma-\sigma}}{\Gamma(\sigma+k)} \int_{\xi}^{1} (x-\xi)^{\sigma+k-1} D_x^{k}\{(1-x)^{\gamma} g(x)\}\, dx, \tag{23}$$

where $y<\xi<1$, $\gamma-k>1$, and $\sigma+k>0$, $k$ being a non-negative integer.

For $\rho = \mu - \beta$ and $\sigma = \lambda - \gamma$, the first members of equations (21) and (23) are identical, provided that

$$\alpha + \beta = \gamma + \delta = \lambda + \mu \,, \tag{24}$$

and by appealing to the orthogonality property (13), we finally obtain the desired solution of the dual series equations (10) and (11) in the form:

$$A_n = \frac{(n+l)!\,(\alpha+\beta+2n+2l+1)\,\Gamma(\alpha+\beta+n+l+1)}{2^{\alpha+\beta+1}\,\Gamma(\lambda+n+l+1)\,\Gamma(\mu+n+l+1)} \cdot \left[\frac{1}{\Gamma(\mu-\beta+m)}\int_{-1}^{y}(1-\xi)^{\lambda}\,P_{n+l}^{(\lambda,\mu)}(\xi)\,F(\xi)\,d\xi + \frac{(-1)^k}{\Gamma(\lambda-\gamma+k)}\int_{y}^{1}(1+\xi)^{\mu}\,P_{n+l}^{(\lambda,\mu)}(\xi)\,G(\xi)\,d\xi\right], \tag{25}$$

where, for convenience,

$$F(\xi) = D_{\xi}^{m}\left\{\int_{-1}^{\xi}(1+x)^{\beta}\,(\xi-x)^{\mu-\beta+m-1}f(x)\,dx\right\}, \tag{26}$$

$$G(\xi) = \int_{\xi}^{1}(x-\xi)^{\lambda-\gamma+k-1}\,D_x^k\{(1-x)^{\gamma}g(x)\}\,dx\,, \tag{27}$$

$k, l, m, n$ are non-negative integers, and in addition to the parametric constraints in (12) and (24), we require that

$$\lambda > \gamma - k > -1 \quad , \quad \mu - \beta + m > 0\,. \tag{28}$$

## 3. Concluding remarks

Obviously, when $l = 0$, $\lambda = \alpha + \beta - \mu$ and $\gamma = \alpha + \beta - \delta$, the dual series equations (10) and (11) would correspond to Dwivedi's equations (4) and (5), and hence also to Thakare's equations (7) and (8) if we replace the parameter $\mu$ by $\mu + \beta$. Thus, under the special cases just stated, our solution given by equation (25) would readily yield the results obtained by these earlier writers. And indeed, it will lead to the solution of the dual series equations considered by Noble [4] when we further set $\delta = \beta$, and to that of Srivastav's equations [7] if, in addition to the aforementioned parametric constraints, we set $\mu = \beta + \frac{1}{2}$.

We should like to conclude by remarking further that (by appropriately specializing our equations (34) and (35) of Part II of our work) we can easily obtain the values of the dual series, considered by Dwivedi [1] and Thakare [10], on the intervals over which their values are not already specified. As a matter of fact, this important aspect of the analysis to be presented in Part II was not given by either of these earlier writers.

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