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Robinson consistency theorem in soft model theory

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 15 dicembre 1979

Presiede il Presidente della Classe Antonio Carrelli

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Logica matematica. — *Robinson consistency theorem in soft model theory*. Nota di Daniele Mundici, presentata (*) dal Socio G. Zappa.

Riassunto. — Si enunciano alcuni risultati provati dall'autore negli articoli citati nella bibliografia, alcuni dei quali in corso di stampa su Zeit für Math. Logik, Archiv für Math. Logik, Trans. AMS e sul Journal of Symbolic Logic. Tali risultati collegano le nozioni di interpolazione e compattezza nell'ambito della teoria astratta dei modelli, al teorema di consistenza di Robinson, oppure a nozioni puramente algebriche. In particolare, in ogni logica il teorema di consistenza di Robinson è equivalente al teorema di interpolazione di Craig insieme alla compattezza; quest'ultima, a sua volta, equivale alla proprietà « JEP » di immersione congiunta. Altri risultati riguardano il terzo e quarto problema di H. Friedman e le estensioni della logica col quantificatore « ci sono incontabilmente molti ».

Introduction

Robinson consistency theorem is important in abstract model theory, and has been studied extensively in [5], [6] and, by the author, in [8]–[14]. In [10] and [11] and, independently, in [6], such identities have been established as:

Robinson consistency = compactness + Craig interpolation and:

Compactness = JEP (i.e. the Joint Embedding Property of L-elementary embeddings), in any logic.

(*) Nella seduta del 15 dicembre 1979.

The techniques developed for the study of the Robinson consistency theorem have also proved useful in the investigation, in [9] and [12], of H. Friedman's third and fourth problem in [4], and for the study of extensions of $L(Q_1)$, where Q_1 is the quantifier "there exist uncountably many" (see [8]). For the notions and notations used in abstract model theory, see [2], [3], [1], [7] and [5].

The definition of Robinson consistency theorem is naturally generalized from the familiar notion of first order logic, to the case of an arbitrary logic.

Statement of the results

Our first two theorems, proved in [14], relate countable compactness to Robinson consistency theorem:

THEOREM 1. *Let in logic* L *Robinson consistency theorem hold*; *assume further that* L *is not countably compact*; *then for any single-sorted structure* $\mathfrak{M}$ *with* $|\mathfrak{M}| < \omega_\omega$ *we have*

$$\mathrm{mod}_L\, \mathrm{th}_L\, \mathfrak{M} = I\mathfrak{M} \qquad \text{(i.e. } \textit{the isomorphism class of } \mathfrak{M}\text{)}.$$

THEOREM 2. *Assume that* $2^\omega < 2^{\omega_n}$ *for some* $n \in \omega$; *let in logic* $L = L_{\omega\omega}(Q^i)_{i \in \omega}$ *Robinson consistency theorem hold*; *then in* L *Craig interpolation theorem holds as well*, *and* L *is countably compact.*

Remark. Theorem 2 above improves [6, Th. 6, 11], where the same result is obtained by making use of an additional "Feferman-Vaught" property which L is assumed to have.

The following theorem, proved in [8], deals with logics extending $L(Q_1)$; notice that it does not use any special set-theoretical assumption:

THEOREM 3. *If* $L \geq L(Q_1)$ *satisfies Robinson consistency theorem*, *then*:

(i) L *is not axiomatizable*, *nor countably compact*, *and*

(ii) *for any single sorted structure* $\mathfrak{M}$ *with* $|\mathfrak{M}| < \omega_\omega$, *we have*

$$\mathrm{mod}_L\, \mathrm{th}_L\, \mathfrak{M} = I\mathfrak{M}.$$

Notice that (ii) follows immediately from (i) in the light of Theorem 1.

From Theorems 3 and 1 one gets the following.

THEOREM 4. *Assume that* $2^\omega < 2^{\omega_n}$ *for some* $n \in \omega$; *then no countably generated extension* L *of* $L(Q_1)$ *satisfies Robinson consistency theorem.*

THEOREM 5. *Assume the Continuum hypothesis*, *or even that* $\omega_n = 2^\omega$ *for some* $n \in \omega$; *then if* $L \geq L(Q_1)$ *satisfies Robinson consistency theorem*, *the system of real numbers is characterizable by its complete theory in* L.

The following theorems were proved in [13] and [11] and, independently, in [6] (by assuming that there is no uncountable measurable cardinal and that logic L has an "occurrence number"). Let $\neg L^{\mu}$ mean that there is no inner model with a measurable cardinal $> \omega$. Then we have:

THEOREM 6. *(Assuming* $\neg O^{\#}$ *or even* $\neg L^{\mu}$*). Let in logic* L *the class of sentences of type* τ *be a set if so is* τ*; then* L *satisfies Robinson consistency theorem iff* L *is compact and satisfies Craig interpolation theorem.*

THEOREM 7. *Under the same hypotheses of Theorem* 6, L *is compact iff the* L*-elementary embeddings have the Joint Embedding Property (JEP).*

Our final results are about H. Friedman's problems in [4]. His third problem, of finding logics strictly between $L_{\infty\omega}$ and $L_{\infty\infty}$ satisfying Craig interpolation theorem is still unsolved; but in [12] we proved the following.

THEOREM 8. *(Assuming* $\neg O^{\#}$, *or* $\neg L^{\mu}$*). Let* $k \geq \omega$*; then no logic between* $L_{\infty\omega}$ *and* $L_{\infty k}$ *satisfies Craig interpolation theorem, nor Robinson consistency theorem.*

Also H. Friedman's fourth problem of finding proper extensions of first order logic with compactness and interpolation is still unsolved; a related result is the following theorem, which immediately descends from Theorem 1 and Lindstrom theorem:

THEOREM 9. *Let* L *be a countably generated logic satisfying Robinson consistency theorem; assume that each consistent theory* T *with* $|T| \leq \omega$ *has a model of cardinality* $\leq \omega$. *Then* $L \sim L_{\omega\omega}$.

By making use of an algebraic result due to the author in [9], we also have:

THEOREM 10. *(Assuming* $\neg O^{\#}$, *or* $\neg L^{\mu}$*). Let* $L = L_{\omega\omega}(Q^i)_{i \in I}$, *with* I *a set, satisfy Robinson consistency theorem; assume that* L *also satisfies Robinson consistency theorem upon restriction to the class of countable structures of finite type. Then*

$$\mathfrak{M} \equiv_L \mathfrak{N} \quad \textit{iff} \quad \mathfrak{M} \equiv \mathfrak{N}$$

(for any two countable structures $\mathfrak{M}$ *and* $\mathfrak{N}$*).*

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