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Quasi-linear systems and waves in thermoviscous fluiddynamics

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Fisica matematica. — *Quasi-linear systems and waves in thermo-viscous fluiddynamics* (*). Nota di ANGELO MORRO, presentata (**) dal Socio D. GRAFFI.

RIASSUNTO. — Si considera un fluido viscoso e conduttore di calore il cui comportamento è descritto mediante variabili nascoste. Si mostra che le corrispondenti equazioni di evoluzione, insieme alle usuali equazioni di bilancio, individuano un sistema iperbolico di equazioni quasi-lineari. Tra le onde ammesse dalla teoria si esaminano in dettaglio le onde trasversali propagantisi in regioni in equilibrio; tali onde risultano eccezionali.

1. INTRODUCTION

Elasticity, viscosity, and heat conduction are outstanding features exhibited by the behaviour of material bodies. It is the aim of the constitutive theories to provide appropriate schemes whereby these features are accounted for so as to make the theoretical predictions as close as possible to the experimental results. Within the framework of fluiddynamics a prominent constitutive model is the *Maxwell fluid* where viscosity involves a relaxation time [1]. An improvement of the Maxwell fluid is supplied by the *Maxwellian materials* investigated by Coleman, Greenberg and Gurtin [2]. Also, in connection with heat conduction models, we know the Maxwell-Cattaneo equation [3, 4]; to my mind Cattaneo's paper [4] marks the beginning of the wide activity in the context of constitutive theories compatible with wave propagation.

The recent scientific literature bears evidence of the hidden variables as a very fruitful tool in constitutive theories [5–7]. For instance, the question of wave propagation in heat-conducting viscous fluids has been given a satisfactory solution by having recourse to fluids with hidden variables [8–10]. This fact is hardly surprising inasmuch as account for hidden variables amounts to describing in a suitable way relaxation phenomena actually occurring in fluids.

To get further insights into the hidden variable approach this note investigates a model of fluid with hidden variables—sec. 2—from the standpoint of quasi-linear systems of equations. It is shown that if the present values of the hidden variables vanish then the corresponding system of

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equations turns out to be hyperbolic—sec. 3. This warrants the study of wave—weak discontinuity—propagation into a region at equilibrium; a conspicuous result of this study, namely the exceptionality of the transverse waves, is delivered in section 4.

2. ESSENTIALS OF HIDDEN VARIABLES IN FLUIDDYNAMICS

Henceforth fluids with hidden variables are specified by C^2 response functions

$$\sigma = \sigma(\theta, \rho, \alpha', \alpha'') \tag{2.1}$$

and by the linear evolution equations

$$\begin{aligned} \dot{\alpha}' &= \frac{1}{\tau'}(\nabla\theta - \alpha'), \qquad \tau' > 0, \\ \dot{\alpha}'' &= \frac{1}{\tau''}(\mathbf{D} - \alpha''), \qquad \tau'' > 0, \end{aligned} \tag{2.2}$$

where $\alpha' \in R^3$ and $\alpha'' \in \mathrm{Sym}(R^3, R^3)$ are the hidden variables while the other notations are the usual ones [9]. The strict counterparts of Navier-Stokes' and Fourier's laws are obtained by choosing the free energy ψ as

$$\psi(\theta, \rho, \alpha', \alpha'') = \Psi(\theta, \rho) + \frac{1}{\rho}\left\{\frac{\varkappa\tau'}{2\theta}\alpha'\cdot\alpha' + \mu\tau''\alpha'' : \alpha'' + \tfrac{1}{2}\lambda\tau''(tr\,\alpha'')^2\right\}.$$

Indeed compatibility with the second law of thermodynamics in the form of the Clausius-Duhem inequality implies that [9]

$$\mu \geq 0 \quad , \quad 3\lambda + 2\mu \geq 0 \quad , \quad \varkappa \geq 0,$$

and

$$\begin{aligned} \eta &= -\Psi_\theta + \frac{\varkappa\tau'}{2\rho\theta^2}\alpha'\cdot\alpha', \\ \mathbf{T} &= -p\mathbf{I} + 2\mu\alpha'' + \lambda(tr\,\alpha'')\mathbf{I} \quad , \quad \boldsymbol{q} = -\varkappa\alpha'. \end{aligned} \tag{2.3}$$

The pressure p is defined as $p = \rho^2\psi_\rho$; the subscripts θ, ρ denote partial differentiations.

Before examining some properties of the present hidden variable approach it is worth pointing out that evolution equations like (2.2) are well known in the literature. For example, letting $\alpha'' = \mathbf{T}/2\mu\tau''$ eq. $(2.2)_2$ becomes just the constitutive equation of the Maxwell fluid. Also, letting $\alpha' = -\boldsymbol{q}/\varkappa$ eq. $(2.2)_1$ becomes the Maxwell-Cattaneo equation. Moreover, equations accounting for relaxation phenomena have a structure like (2.2);

such is the case, for example, of Graffi's equations for the mechanical behaviour of ionised gases [11] and of Coleman-Gurtin's equations for the thermal behaviour of gases [12]. Yet, the very advantage of the hidden variable approach over other continuum approaches is that the evolution equations are incorporated into a systematic scheme leading to constitutive equations which are automatically consistent with thermodynamics.

3. Hyperbolicity of the quasi-linear system

On adopting the usual notations [9] the balance equations read

$$\begin{aligned} &\dot{\rho} + \rho\nabla\cdot\mathbf{V} = 0\,, \\ &\rho\dot{\mathbf{V}} - \nabla\cdot\mathbf{T} = \mathbf{0}\,, \\ &\rho\dot{e} - \mathbf{T} : \mathbf{D} + \nabla\cdot\boldsymbol{q} = 0\,. \end{aligned} \tag{3.1}$$

Introduce now a fixed orthonormal basis $\boldsymbol{e}_1 , \boldsymbol{e}_2 , \boldsymbol{e}_3$ and the ordered array

$$u^{\mathrm{A}} = (\rho\,, \mathrm{V}_3\,, \mathrm{V}_1\,, \mathrm{V}_2\,, \theta\,, \alpha'_3\,, \alpha'_1\,, \alpha'_2\,, \alpha''_{33}\,, \alpha''_{11}\,, \alpha''_{22}\,, \alpha''_{13}\,, \alpha''_{23}\,, \alpha''_{12})\,.$$

Then, on appealing to the constitutive equations (2.3), a direct calculation allows us to write the full set of the balance equations (3.1) and of the evolution equations (2.2) as a first-order quasi-linear system, namely

$$u^{\mathrm{A}}{}_{,4} + (a^i)^{\mathrm{A}}{}_{\mathrm{B}}\, u^{\mathrm{B}}{}_{,i} + b^{\mathrm{A}} = 0 \;{}^{(1)}\,. \tag{3.2}$$

The array b^{A} contains terms dependent on $\rho , \theta , \alpha', \alpha''$ but not on their derivatives; next developments are unaffected by the particular structure of b^{A} and, meanwhile, they involve the 14×14 matrices $(a^i)^{\mathrm{A}}{}_{\mathrm{B}}\,, i = 1 , 2 , 3,$ only through the matrix (3.3).

Let $f(\boldsymbol{x} , t) = 0$ be a surface in space-time and denote by $\boldsymbol{n} = \nabla f / | \nabla f |$ its unit normal. For $f = 0$ to be a characteristic surface the function f must satisfy the determinantal equation

$$| (a^i)^{\mathrm{A}}{}_{\mathrm{B}}\, f_{,i} + \delta^{\mathrm{A}}{}_{\mathrm{B}}\, f_{,4} | = 0\,.$$

In conjunction with the general theory [13, 14] we can say that the system (3.2) is hyperbolic if all the eigenvalues of $(a^i)^{\mathrm{A}}{}_{\mathrm{B}}\, n_i$ are real and if $(a^i)^{\mathrm{A}}{}_{\mathrm{B}}\, n_i$ possesses a complete set of linearly independent eigenvectors. Without any loss of generality set $\boldsymbol{e}_3 = \boldsymbol{n}$; then the matrix $(a^i)^{\mathrm{A}}{}_{\mathrm{B}}\, n_i$ takes the simplified form

(1) Latin small (capital) indices haves the range $1 , 2 , 3\,(1 , 2 ,\cdots, 14)$ and repeated indices are summed over this range. A subscript preceded by a comma indicates differentiation with respect to the corresponding x-coordinate ; $x_4 \equiv t$.

$$
(3.3)\quad (a^i)^{\mathrm{A}}{}_{\mathrm{B}}\, n_i = \begin{vmatrix}
V_n & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{p_\rho}{\rho} & V_n & 0 & 0 & \frac{p_\theta}{\rho} & -\frac{\varkappa\tau'}{\rho\theta}\alpha'_n & -\frac{\varkappa\tau'}{\rho\theta}\alpha'_1 & -\frac{\varkappa\tau'}{\rho\theta}\alpha'_2 & \varepsilon_3 & \varepsilon_1 & \varepsilon_2 & -\frac{4\mu\tau''}{\rho}\alpha''_{13} & -\frac{4\mu\tau''}{\rho}\alpha''_{23} & -\frac{4\mu\tau''}{\rho}\alpha''_{12} \\
0 & 0 & V_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2\mu}{\rho} & 0 & 0 \\
0 & 0 & 0 & V_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2\mu}{\rho} & 0 \\
0 & -\frac{\rho\eta_\rho}{\eta_\theta} & 0 & 0 & V_n+\zeta & -\frac{\varkappa}{\rho\theta\eta_\theta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{\tau'} & V_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & V_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & V_n & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{\tau''} & 0 & 0 & 0 & 0 & 0 & 0 & V_n & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_n & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_n & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2\tau''} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_n & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2\tau''} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_n & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_n
\end{vmatrix}
$$

where

$$V_n = \mathbf{V}\cdot \boldsymbol{n} \quad , \quad \alpha'_n = \alpha'\cdot \boldsymbol{n} \quad , \quad \zeta = 2\,\varkappa\alpha'_n/\rho\theta^2\,\eta_\theta\,,$$

$$\varepsilon_1 = -(2\,\mu\tau''/\rho)\,\alpha''_{11} - (\lambda/\rho)\,(1+\tau''\,tr\,\alpha'')\,,$$

$$\varepsilon_2 = -(2\,\mu\tau''/\rho)\,\alpha''_{22} - (\lambda/\rho)\,(1+\tau''\,tr\,\alpha'')\,,$$

$$\varepsilon_3 = -(2\,\mu/\rho)\,(1+\tau''\,\alpha''_{33}) - (\lambda/\rho)\,(1+\tau''\,tr\,\alpha'')\,.$$

To determine all the eigenvalues of $(a^i)^A{}_B\, n_i$ in the general case is a prohibitive task. Conversely it is a simple matter to see that if the present values of the hidden variables α', α'' are unrestricted then the system (3.2) may be non-hyperbolic [15]. In view of this I have elected to evaluate the eigenvalues of (3.3) under the assumption that the present values of the hidden variables vanish (2). In such a case a straightforward calculation yields the following eigenvalues

$$c_0 = V_n \quad , \quad m = 6\,, \qquad c_f^+ = V_n + U_f \quad , \quad m = 1\,,$$

$$c_T^+ = V_n + \left(\frac{\mu}{\rho\tau''}\right)^{\frac{1}{2}} \quad , \quad m = 2\,, \qquad c_f^- = V_n - U_f \quad , \quad m = 1\,,$$

$$c_s^+ = V_n + U_s \quad , \quad m = 1\,,$$

$$c_T^- = V_n - \left(\frac{\mu}{\rho\tau''}\right)^{\frac{1}{2}} \quad , \quad m = 2\,, \qquad c_s^- = V_n - U_s \quad , \quad m = 1\,,$$

m being the multiplicity of the eigenvalue. The fast and slow speeds $U_f, U_s > 0$ are given by

$$U_f = \tfrac{1}{2}\{\phi + (\phi^2 - 4\,\omega)^{\frac{1}{2}}\} \quad , \quad U_s = \tfrac{1}{2}\{\phi - (\phi^2 - 4\,\omega)^{\frac{1}{2}}\}\,,$$

where

$$\phi = \hat{p}_\rho + \frac{\varkappa}{\rho\theta\,\eta_\theta\,\tau'} + \frac{2\,\mu+\lambda}{\rho\tau''} \quad , \quad \omega = \frac{\varkappa}{\rho\theta\,\eta_\theta\,\tau'}\left(\hat{p}_\rho + \frac{2\,\mu+\lambda}{\mu\tau''}\right),$$

the symbol $\hat{p}_\rho$ denoting the derivative of p with respect to ρ at constant entropy. Of course, as we should expect, the previous set of eigenvalues collapses to

$$c_0 = V_n \quad , \quad c^+ = V_n + (\hat{p}_\rho)^{\frac{1}{2}} \quad , \quad c^- = V_n - (\hat{p}_\rho)^{\frac{1}{2}}$$

when heat conduction and viscosity are disregarded.

In spite of the fact that only seven distinct eigenvalues occur, it is a routine matter to show that fourteen linearly independent real eigenvectors can be found. Hence we conclude that if $\alpha' = \mathbf{0}$, $\alpha'' = \mathbf{0}$ then the system (3.2) is hyperbolic.

(2) The physical motivation of this choice will become apparent shortly.

4. TRANSVERSE WAVES

The speed of displacement of a wave front $f(\boldsymbol{x}, t) = 0$ must coincide with any of the eigenvalues (3.4). The admissible jump discontinuity of u^A across $f = 0$ is given, to within a multiplicative constant, by the corresponding right eigenvector r^A of $(a^i)^A{}_B\, n_i$. On the other hand, according to eq. (2.2), if the fluid has been held at equilibrium ($\nabla\theta = \mathbf{0}$, $\mathbf{D} = \mathbf{0}$) up to time t then the hidden variables too vanish up to time t. In view of these facts we can conclude that the set of eigenvalues (3.4) provides the admissible speeds of displacement of waves propagating into a region at equilibrium.

Look now at the propagation modes associated with plane waves propagating in the fluid at the speeds $c_T^{\pm} = V_n \pm (\mu/\rho\tau'')^{\frac{1}{2}}$. Consistently with the condition $m = 2$, for any eigenvalue c_T^+ and c_T^- we find two linearly independent right eigenvectors $(r_1^A)^+, (r_2^A)^+$ and $(r_1^A)^-, (r_2^A)^-$. If we let $\alpha' = \mathbf{0}$, $\alpha'' = \mathbf{0}$ they can be written in the reduced forms

$$
(4.1)\qquad
\begin{aligned}
(r_1^A)_0^{\pm} &= \left(0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, \mp \tfrac{1}{2}\left(\frac{\rho}{\mu\tau''}\right)^{\frac{1}{2}}, 0, 0\right),\\
(r_2^A)_0^{\pm} &= \left(0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, \mp \tfrac{1}{2}\left(\frac{\rho}{\mu\tau''}\right)^{\frac{1}{2}}, 0\right),
\end{aligned}
$$

which are easily seen to correspond to transverse waves polarised in the directions $\boldsymbol{e}_1, \boldsymbol{e}_2$, respectively. This is so because the third (fourth) and the twelfth (thirteenth) components are proportional to the jumps of $\nabla_n V_1$ ($\nabla_n V_2$) and $\nabla_n \alpha_{13}''$ ($\nabla_n \alpha_{23}''$), respectively. Further, one glance at (4.1) allows us to assert that, like the Alfvén waves, the transverse waves in heat-conducting viscous fluids are hydrodynamic homothermal waves, that is to say $[\nabla_n \rho] = 0$, $[\nabla_n \theta] = 0$.

A noteworthy property of the transverse waves concerns the evolution of the discontinuity. Indeed a routine calculation proves that the transverse waves propagating into a region at equilibrium are exceptional; this generalises the analogous result found by Bampi and myself [15] in connection with viscous fluids ($\varkappa = 0$). To derive this result observe first that the left eigenvectors l_A of $(a_i)^A{}_B\, n_i$ pertaining to the eigenvalues $c_T^{\pm}$ may be written as

$$
\begin{aligned}
(l_A^1)^{\pm} &= \left(0, 0, \tfrac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, \mp \left(\frac{\mu\tau''}{\rho}\right)^{\frac{1}{2}}, 0, 0\right),\\
(l_A^2)^{\pm} &= \left(0, 0, 0, \tfrac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, \mp \left(\frac{\mu\tau''}{\rho}\right)^{\frac{1}{2}}, 0\right),
\end{aligned}
$$

even though $\alpha' \neq \mathbf{0}$, $\alpha'' \neq \mathbf{0}$. Accordingly the quantity [14]

$$\mathrm{N} = \{\mathrm{D_Q}\,(l_\mathrm{A}\,(a^i)^\mathrm{A}{}_\mathrm{B})\}_0\, f_{,i}\,(r^\mathrm{Q})_0\,(r^\mathrm{B})_0 + \{\mathrm{D_Q}\, l_\mathrm{B}\}_0\, f_{,4}\,(r_\mathrm{Q})_0\,(r^\mathrm{B})_0$$

vanishes identically and this in turn implies the exceptionality of the transverse waves.

It is worth emphasising that the existence of transverse waves is not exclusive consequence of the present hidden variable model. For example, besides an alternative to the present model outlined in [9] § 5, as shown by Franchi [16] also a Maxwell-like fluid model [17] accounts for the existence of transverse hydrodynamic homothermal waves.

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