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Fisica matematica. — *On the contribution of heat flux to the propagation velocity of Relativistic shock waves in thermo-elastic Bodies* (*). Nota I di ALDO BRESSAN, presentata (**) dal Corrisp. G. GRIOLI.

RIASSUNTO. — Si studiano onde d'urto termomeccaniche, (precisamente T–η–onde d'urto) in corpi elastici (o fluidi non viscosi) in una teoria di relatività ristretta o generale, includente il tensore termodinamico di C. Eckart (cfr. [2]). La velocità di propagazione V di queste è calcolata in vari casi, almeno a meno di termini d'ordine 2 (rispetto a $1/c$ ove c è la velocità della luce nel vuoto). A questo scopo è essenziale usare, per esempio, un certo postulato di carattere generale, il quale è compatibile con un'ipotesi di solito fatta implicitamente. Nel caso più generale V dipende da certi rapporti fra parametri di discontinuità e loro derivate. Questi rapporti spariscono in casi speciali importanti concernenti i solidi, e in ogni caso riguardante i fluidi. In particolare è posta in evidenza la dipendenza di V dal flusso di calore.

1. INTRODUCTION

The present work, divided into two notes, is based on the theory $\mathscr{T}$ of (special or general) relativity, of Eckart's type, presented e.g. in [2]; and it deals with a T-η-shock wave σ_t [N. 4] travelling in a thermo-elastic body $\mathscr{C}$, so that by definition the position gradient α^ρ_L, (4-velocity u^ρ), absolute temperature T, and specific entropy η have first order discontinuities $[\alpha^\rho_L]$ to $[\eta]$ across σ_t, while position, the metric tensor $g_{\alpha\beta}$, and its first and second partial derivatives are continuous across σ_t. The body $\mathscr{C}$ is regarded, first, as a fluid [NN. 1–6] and then, more generally, also as an elastic body [NN. 7–9].

This work is compatible with the assumption $[\eta] = 0$, usually made when the heat flux vector vanishes ($q^\alpha \equiv 0$). However it is not restricted to it. More generally $[\eta]$ is postulated to be a certain (constitutive) linear function of $[u^\alpha]$ (cfr. Post. 6.1 for fluids and Post. 8.1 for elastic bodies). This is reasonable because, since (generally) $[T] \neq 0$ across σ_t, by the Fourier law the heat that at the "instant" t crosses the material surface $\bar{\sigma}^*$ occupied by σ_t at $\bar{t}$, has the expression $\infty \cdot 0$, so that it may have any finite value.

Experiments say that, even if $[\eta] = 0$ cannot be postulated, $[\eta]$ must be very small with respect to e.g. $[T]$. Since relativistic corrections also are very small, the afore-mentioned general postulates (compatible with $[\eta] \neq 0$) are more interesting than their analogues in classical physics

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(**) Nella seduta del 14 giugno 1979.

(cf. [4]). However to consider their special case $[\eta] = 0$ is important also in $\mathscr{T}$, e.g. for comparison with previous results.

The main aim of this work is to calculate the propagation speed V of σ_t either exactly or up to terms of order 4 (in $1/c$, where c is the speed of light in vacuum). If $\mathscr{C}$ is an elastic body, in the general case certain ratios among discontinuity parameters and their spatial derivatives occur in our expressions for V [N. 9]; however they disappear in most important cases. Such ratios never appear when $\mathscr{C}$ is a (non-viscous) fluid [N. 6]. For these fluids V can be determined up to terms of order 6 (cf. (6.6)); furthermore, if the heat flux vector q^α is orthogonal to σ_t, an exact polynomial equation of the third order in V holds (cf. (5.13)) with $|\boxed{6}| = 0$.

The relativistic corrections to V, of order 2, for non-viscous fluids, in the general case are put in evidence by (6.5). Among them the contribution of q^α is generally $\neq 0$. Similar corrections of the same order, seem to exist for elastic bodies in the general case (cf. (9.4–6)); and the relativistic contributions of order 2, in these results, due to q^α are satisfactory because of the same order are the relativistic corrections found so far (cf. e.g. [2, § 66]).

The aforementioned results contain new features also for $q^\alpha \equiv 0$ (and when $\mathscr{C}$ is a fluid) in that $[\eta]$ is allowed to be $\neq 0$. The secular equation (9.8) in V, for T-η-shock waves in elastic bodies with $q^\alpha \equiv 0 \equiv [\eta]$ (1), shows that V coincides with its well known analogue for acceleration waves (cf. [2, § 66]).

The present work is a relativistic extension of the first part of [4] belonging to classical physics; hence it is also related with the analogue [3] of [4] for acceleration waves; and its motivations appear strengthened by considerations made in [3] or [4]. However the present work is independent of these papers, not yet published or appeared.

* * *

Some preliminaries based on [2] concern space-time and discontinuity waves from the Eulerian and Lagrangian points of view [NN. 2, 3, 5] (2). A natural analogue of the global balance equations for energy and momentum is postulated in general relativity and from it the corresponding Kotchine theorems on σ_t are deduced [N. 4]. Some thermo-mechanic considerations [N. 5] allow us to find the afore-mentioned expressions for V in connection with viscous fluids. Thermo-mechanic equations

(1) The autor has not seen such a result on shock waves (in elastic solids with $q^\alpha \equiv 0 \equiv [\eta]$) in the literature.

(2) The relativistic relation (5.4) between [V] and the discontinuity $[cu^\rho N_\rho]$ of the normal velocity of $\mathscr{C}$, as well as its proof, has been written only because in the present frame work the proof is very short.

for shock waves in elastic solids are deduced in N. 7. With the aid of Post 8.1 they allow us to determine V in the general elastic case and in some important special cases, in various ways.

2. PRELIMINARIES ON SPACE-TIME FROM THE EULERIAN AND LAGRANGIAN POINTS OF VIEW

For phenomena in the relativistic space time S_4 we use the notations of [2]. By x^ρ we denote the co-ordinates of the typical event point $\mathscr{E}$ (of S_4) in the admissible frame or co-ordinate system (x), and by [(3)]

$$(2.1) \qquad ds^2 = -g_{\rho\sigma}\, dx^\rho\, dx^\sigma \qquad \text{with} \quad g_{00} < 0$$

the space time metric in S_4.

Let $\mathscr{C}$ be a 3-dimensional continuous body, thought of as a set of matter points. Let D denote differentiation along world-lines and set

$$(2.2) \qquad u^\alpha = \frac{Dx^\alpha}{Ds} \quad , \quad A^\alpha = \frac{Du^\alpha}{Ds} \quad , \quad \overset{1}{g}_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta \quad , \quad T^{\cdots}_{\cdots/\overset{1}{\alpha}} = T^{\cdots}_{\cdots/\beta}\, \overset{1}{g}{}^{\beta}_{\alpha}.$$

Let $d\mathscr{C}$ be an element of $\mathscr{C}$ containing the matter point P^*. Let $d\mathscr{C}$ $[c^{-2}\rho\, dC]$ be the actual proper volume [gravitational mass] of $d\mathscr{C}$, and let dC^* $[k^*\, dC^*]$ be its analogue in the reference state Σ^* (of $\mathscr{C}$). Denote by k the actual density of the conventional (or proper reference) mass $k^*\, dC^*$ (of $d\mathscr{C}$) (cf. [2, p. 54]). Then the specific internal energy w of $\mathscr{C}$ at P^* can be defined briefly by

$$(2.3) \qquad \rho = k\,(c^2 + w) \qquad (k\, dC = k^*\, dC^*)\,.$$

By " specific " we mean both proper and per unit proper reference mass.

The preceding considerations of a Eulerian type are fit to treat e.g. fluids. For dealing with solids, in particular (thermo-)elastic materials, the Lagrangian point of view is convenient. So we introduce a reference configuration (cf. [2, p. 139]) taken by $\mathscr{C}$ together with the state Σ^*, along a reference process $\mathscr{P}^*$, on the hypersurface $y^0 = 0$, where (y) is an admissible co-ordinate system for the version S_4^* of S_4 related to $\mathscr{P}^*$. Then the L-th co-ordinate of the intersection of this hypersurface with the world-line of the (arbitrary) matter point P^* of $\mathscr{C}$ is called the L-th *material co-ordinate* of P^* (cf. ftn. (3)). The (strictly positive) *material metric* ds^{*2} (relative to $\mathscr{P}^*$ and (y)) is defined by

$$(2.4) \qquad ds^{*2} = \overset{*}{a}_{LM}\, dy^L\, dy^M \qquad \text{with} \quad \overset{*}{a}_{LM}(y^1, y^2, y^3) = \overset{1*}{g}_{LM}(0, y^1, y^2, y^3)\,,$$

(3) Greek [Latin] indices run from 0 [1] to 3. Einstein's convention on free or bound indices is used –e.g. $T^{\rho\sigma}{}_{/\sigma} = 0$ stands for: $\sum_{\sigma=0}^{4} T^{\rho\sigma}{}_{/\sigma} = 0\ (\rho = 0, \cdots, 3)$.

where $\overset{*}{g}_{\sigma\rho}$ is the space-time metric tensor associated with $\mathscr{P}^*$. The motion $\mathscr{M}$ of $\mathscr{C}$ in Σ_4 has a representation

$$x^\rho = x^\rho(t, y^L) \equiv x^\rho(t, \boldsymbol{y}) \qquad [t = \hat{t}(\boldsymbol{x})], \tag{2.5}$$

which is determined only up to a substitution of the time parameter $\hat{t}$ (cf. [2, p. 140]). We also introduce the position gradient α^ρ_L, the (first right) Cauchy-Green tensor C_{LM} and the deformation tensor ε_{LM} (cf. [2, § 53];

$$\alpha^\rho_L = \overset{\perp}{g}{}^\rho_\sigma\, x^\sigma_{,L} \quad , \quad \overset{*}{a}_{LM} + 2\,\varepsilon_{LM} = C_{LM} = \overset{\perp}{g}_{\rho\sigma}\, x^\rho_{,L}\, x^\sigma_{,M} \qquad \left(f_{,L} = \frac{\partial f}{\partial y^L}\right); \tag{2.6}$$

furthermore the volume ratio $\mathscr{D} = dC/dC^*$ and the spatial inverse $\mathscr{D}^{-1}\gamma^L_\rho$ of α^ρ_L (cf. [2, $(56.4)_1$, $(56.9)_1$, and (56.15)])

$$\left\{\begin{array}{lll} \mathscr{D}^2 = \left(\dfrac{dC}{dC^*}\right)^2 = \dfrac{1}{a^*}\det\|C_{LM}\| & , & \gamma^L_\rho\, \alpha^\sigma_L = \mathscr{D}\overset{\perp}{g}{}^\sigma_\rho\,, \\ u^\rho\, \gamma^L_\rho = 0 \quad , \quad \gamma^L_\rho\, \alpha^\rho_M = \mathscr{D}\overset{*}{a}{}^L_M & , & a^* = \det\|\overset{*}{a}_{LM}\|\,. \end{array}\right. \tag{2.7}$$

3. On surfaces moving in space-time

Let σ_3 be a time-like hypersurface travelling in the world-tube $W_{\mathscr{C}}$ of $\mathscr{C}$ and represented by $(3.1)_2$ below.

$$f(\boldsymbol{x}) \equiv f(x^0, \cdots, x^3) = 0 \quad , \quad f^*(t, \boldsymbol{y}) \equiv f[x(t, \boldsymbol{y})] = 0\,. \tag{3.1}$$

Equation $(3.1)_4$ represents the image $\overset{*}{\sigma}_2 = \overset{*}{\sigma}_2(t)$ of σ_3 in the reference configuration (i.e. a 2-dimensional surface moving in a 3-dimensional Riemannian space). Let V $[V_*]$ be the propagation speed of σ_3 $[\overset{*}{\sigma}_2]$ at $\boldsymbol{x}$ $[(t, \boldsymbol{y})]$ and call N_ρ $[\overset{*}{\mathscr{N}}_L]$ the spatial normal [the normal] unit vector of σ_3 $[\overset{*}{\sigma}_2]$ at the same event point (N_ρ is normal to the intersection σ_2 of σ_3 with the hypersurface $x_0 =$ const.). Then (cf. [2, (65.3–6)] remarking that here we give signs to V and V_* and use more general co-ordinates)

$$g N_\rho = f_{/\overset{\perp}{\rho}} \quad , \quad g V = -\, c f_{/\rho}\, u^\rho \qquad (g^2 = \overset{\perp}{g}{}^{\rho\sigma} f_{/\rho} f_{/\sigma}\,,\ g > 0)\,, \tag{3.2}$$

$$g_* \overset{*}{\mathscr{N}}_L = \overset{*}{f}_{/L} \quad , \quad g_* V_* = -\frac{\partial f^*}{\partial t} \qquad (g_*^2 = \overset{*}{f}_{/L} f^{*/L},\ g_* > 0) \tag{3.3}$$

$$\text{for} \quad Dx^0 = Ds = c\,Dt\,,$$

$$g_* \overset{*}{\mathscr{N}}_L = g N_\rho\, \alpha^\rho_L \quad , \quad \frac{V}{V_*} = \frac{g_*}{g} = \gamma' = \gamma\,, \tag{3.4}$$

where

$$\gamma = (\overset{-1}{C}{}^{LM} \overset{*}{\mathscr{N}}_L \overset{*}{\mathscr{N}}_M)^{-\frac{1}{2}} \quad , \quad \gamma' = (C'^{\rho\sigma} N_\rho N_\sigma)^{\frac{1}{2}} \qquad \text{with} \quad C'^{\rho\sigma} = \alpha^\rho_L\, \alpha^{\sigma L}. \tag{3.5}$$

Relations (3.4) can be proved easily by choosing the frame (x) $[(y)]$ to be locally natural and proper [geodesic] [(4)]:

$$(3.6)\qquad \begin{cases} g_{rs} = \delta_{rs} \quad , \quad g_{0\rho} = -\delta_{0\rho} \quad , \quad g_{\rho\sigma,\lambda} = 0 \quad ; \quad u^{\rho} = \delta^{\rho}_{0} \, ; \\ \overset{*}{a}_{\mathrm{LM}} = \delta_{\mathrm{LM}} \quad , \quad \overset{*}{a}_{\mathrm{LM,H}} = 0 \, . \end{cases}$$

As is well known, (x) and $\hat{t}$ can be so chosen that $(3.3)_{5,6}$ hold along the world-line W_{P^*} of every matter point P^* in $\mathscr{C}$.

The absolute derivative $T^{\cdots}_{\cdots;A}$ of a double tensor $T^{\cdots}_{\cdots}$ is now relative to the choice of the time parameter $\hat{t}$ (cf. $(2.5)_3$). For a locally time-orthogonal choice of it (i.e. $\hat{t}\,(x)_{/\rho}$ locally parallel with u_ρ) we have the Lagrangian spatial derivative $T^{\cdots}_{\cdots|A}$, which has tensorial expressions independent of the choice of $\hat{t}$ (cf. e.g. [2, § 53, (53.9)]):

$$(3.7)\qquad T^{\cdots}_{\cdots|A} = T^{\cdots}_{\cdots/\rho}\, a^{\rho}_{A} + T^{\cdots}_{\cdots/A} + \frac{\partial T^{\cdots}_{\cdots}}{\partial t}\, \frac{Dt}{Ds}\, \overset{T}{u}_{A} \qquad (\overset{T}{u}_{A} = u_\rho\, x^{\rho}_{,A}) \, .$$

We use v^1, v^2, briefly $v^{\mathscr{A}}$, as parameters for the equations

$$(3.8)\qquad x^{\rho} = x^{\rho}\,(t\,, v^{\mathscr{A}}) \quad , \quad y^{\rho} = y^{\rho}\,(t\,, v^{\mathscr{A}})$$

of $\sigma_2(t)$ and $\overset{*}{\sigma}_2(t)$ respectively—where $\sigma_2(t)$ $[\overset{*}{\sigma}_2(t)]$ is also represented by $(3.1)_2$ $[(3.1)_4]$ in case $t \equiv x^0$.

For the second fundamental form $b_{\mathscr{A}\mathscr{B}}$ $[\overset{*}{b}_{\mathscr{A}\mathscr{B}}]$ of σ_3 $[\overset{*}{\sigma}_2 = \overset{*}{\sigma}_2(t)]$ the following holds in the Riemannian space S_4 $[\overset{*}{S}_3]$ of metric tensor $g_{\rho\sigma}$ $[\overset{*}{a}_{\mathrm{LM}}]$:

$$(3.9)\qquad b_{\mathscr{A}\mathscr{B}} = N_\rho\, N^{\rho}_{;\mathscr{A}\mathscr{B}} = -x^{\rho}_{;\mathscr{A}}\, N^{\rho}_{;\mathscr{B}} \quad , \quad \overset{*}{b}_{\mathscr{A}\mathscr{B}} = \mathscr{N}^{*}_{L}\, \mathscr{N}^{*L}_{;\mathscr{A}\mathscr{B}} = -y^{L}_{;\mathscr{A}}\, \mathscr{N}^{*}_{L;\mathscr{B}} \, .$$

We accept the conventions

$$(3.10)\qquad 2\,T_{(\rho\sigma)} = T_{\rho\sigma} + T_{\sigma\rho} \quad , \quad 2\,T_{[\rho\sigma]} = T_{\rho\sigma} - T_{\sigma\rho} \, .$$

4. BASIC RELATIVISTIC EQUATIONS FOR THERMO-ELASTIC CONTINUOUS MEDIA. AN ANALOGUE FOR THE BALANCE OF ENERGY AND LINEAR MOMENTUM IN GENERAL RELATIVITY

Let q_α be the (spatial relativistic) heat flux vector, so that for every spatial unit vector N_α, $q^\alpha N_\alpha$ is the energy dW that by heat conduction crosses a material surface of unit normal N_α, per unit proper area and

(4) Assume (3.6). Then, first $g_{*}\mathscr{N}^{*}_{L} = f'_{,L} = f'_{,r}\, x^{r}_{,L} = g N_r\, x^{r}_{,L}$; hence $(3.4)_1$. By (3.6) and $(3.3)_{4-6}$, $c f_{,\rho}\, u^{\rho} = \partial f^{*}/\partial t$, which by $(3.2)_2$ and $(3.3)_2$ yields $(3.4)_2$. By $(3.5)_3$, $g^2_{*} = \delta^{\mathrm{LM}} f_{,L}\, f_{,M} = \delta^{\mathrm{LM}} f_{,r}\, f_{,s}\, x^{r}_{,L}\, x^{s}_{,M} = g^2\, N_r\, N_s\, C'^{rs}$, which by $(3.5)_2$ yields $(3.4)_3$.

Lastly by $(2.6)_3$ $\overset{-1}{C}{}^{\mathrm{LM}} = \delta^{rs}\, y^{L}_{,r}\, y^{M}_{,s}$; furthermore $g N_r = f_{,r} = f_{,L}\, y^{L}_{,r} = g_{*}\, \mathscr{N}^{*}_{L}\, y^{L}_{,r}$. Hence $g^2 = g^2\, \delta^{rs}\, N_r\, N_s = g^2_{*}\, \overset{-1}{C}{}^{\mathrm{LM}}\, \mathscr{N}^{*}_{L}\, \mathscr{N}^{*}_{M}$. By $(3.5)_1$ and $(3.4)_3$ this yields $(3.4)_4$.

Römer time ($q^{\alpha} N_{\alpha} = d^2 W/d\sigma Ds$). We consider the (ordinary size) Fourier coefficient $\mathfrak{c}_{\alpha}^{\rho} = c\mathscr{H}_{\alpha}^{\rho}$ connected with the Fourier-Eckart law (4.1) below, as a functions of $\boldsymbol{y}$, α, and η (cf. [2, (25.2), p. 64])

$$q_{\alpha} = \mathscr{H}_{\alpha}^{\rho}\,(T_{/\rho}^{\pm} + TA_{\rho}) \quad , \quad \mathfrak{c}_{\alpha}^{\rho} = c\mathscr{H}_{\alpha}^{\rho}\,(\boldsymbol{y}\,,\alpha\,,\eta)\,. \tag{4.1}$$

Denoting the (spatial) Euler stress tensor by $X^{\rho\sigma}$ ($= X^{\sigma\rho}$), the conservation equations

$$\mathscr{U}^{\rho\sigma}_{/\sigma} = 0\,, \qquad \text{where} \quad \mathscr{U}^{\rho\sigma} = \rho u^{\rho} u^{\sigma} + X^{\rho\sigma} + 2\,u^{(\rho} q^{\sigma)} \qquad (q^{\rho} u_{\rho} \equiv 0) \tag{4.2}$$

hold in both special and general relativity. Let us choose arbitrarily a vector field Ψ_{α} of class $C^{(1)}$ and a space-time region $\mathscr{V}_4$ whose (3-dimensional) boundary $\Sigma = \mathscr{F}\mathscr{V}_4$ is piecewise of class $C^{(2)}$. Let $n_{\rho}\, d\Sigma$ be the typical element of Σ oriented inward ($n^{\rho} n_{\rho} = \pm 1$ unless $n_{\rho}\, d\Sigma = 0$). Then if, in addition, $\mathscr{U}_{\rho\sigma}$ is continuous in $\mathscr{V}_4$ and (4.2) holds there (nearly everywhere),

$$\int_{\Sigma} \Psi_{\alpha}\, \mathscr{U}^{\alpha\sigma}\, n_{\sigma}\, d\Sigma = -\int_{\mathscr{V}_4} \Psi_{\alpha/\sigma}\, \mathscr{U}^{\alpha\sigma}\, d\mathscr{V}_4 \qquad (\forall \Psi_{\alpha} \in C^{(1)})\,. \tag{4.3}$$

Now fix the index ρ and identify Ψ_{α} with the gradient $x^{\rho}_{/\alpha} = \delta^{\rho}_{\alpha}$ of x^{ρ} in the arbitrary frame (x). Thus $\Psi_{\alpha/\sigma} = -\begin{Bmatrix} \rho \\ \sigma\alpha \end{Bmatrix}$. Then (writing $f_{,\rho}$ for $\partial f/\partial x^{\rho}$)

$$\int_{\Sigma} \mathscr{U}^{\rho\sigma}\, n_{\sigma}\, d\Sigma = \int_{\mathscr{V}_4} \begin{Bmatrix} \rho \\ \alpha\sigma \end{Bmatrix} \mathscr{U}^{\alpha\sigma}\, d\mathscr{V}_4\,; \tag{4.4}$$

hence

$$\int_{\Sigma} \mathscr{U}^{\rho\sigma}\, n_{\sigma}\, d\Sigma = 0 \qquad \text{for} \quad g_{\alpha\beta,\gamma} \equiv 0\,. \tag{4.4'}$$

Thus, in special relativity, (4.4′) holds whenever (x) is a Minkowskian frame. In this case, especially when $\mathscr{V}_4$ is small and the speed of matter inside $\mathscr{V}_4$, relative to (x), is not large, (4.4′) can be easily recognized to express the balance of linear momentum (for $\rho = 1, 2, 3$) and the one of energy for $\rho = 0$. (Indeed the local analogue holds for equation $(4.2)_2$ up to terms of order 2). In classical physics such integral balance relations are assumed to hold also when $\mathscr{V}_4$ contains singular surfaces. Then it is natural to do the same in special relativity. This can be stated in the form (4.3) which is meaningful also in general relativity:

PRINCIPLE *of energy-momentum balance in special or general relativity. Equality* (4.3) *holds for every vector field* Ψ_{α} *of class* $C^{(1)}$ *and every space-time region* $\mathscr{V}_4$ *with* $\mathscr{F}\mathscr{V}_4$ *piecewise of class* $C^{(2)}$.

The region $\mathscr{V}_4$ above may include an (oriented singular) surface σ_3, across which ρ, $X^{\rho\sigma}$, q^{ρ}, and u^{ρ} have discontinuities of the first kind—to be denoted by [ ]. In general relativity we assume

$$[g_{\rho\sigma}] = 0 = [g_{\rho\sigma,\gamma}] \qquad \textit{across} \quad \sigma_3 . \tag{4.5}$$

Under the assumptions above σ_t will be called a *T-η-shock wave*. By Einstein's gravitational equations we generally have $[g_{\rho\sigma,\lambda\mu}] \neq 0$ across shock waves.

Now let us remark that *on a shock wave σ_t we have the relativistic discontinuity relation for energy-momentum balance*

$$[\mathcal{U}^{\rho\sigma}]\, N_\sigma = -\frac{V}{c}\, [\mathcal{U}^{\rho\sigma}]\, u_\sigma \qquad (N_\sigma u^\sigma = 0 \;\; , \;\; N^\sigma N_\sigma = 1) , \tag{4.6}$$

where N_ρ is the unit vector for the spatial section σ_2 of σ_3, oriented positively.

Indeed by reasoning like in [5, p. 527] to prove the classical version of Kotchine's theorem, we identify $\mathcal{V}_4$ with a small cylinder that is symmetric with respect to a (practically) circular small neighborhood $\mathcal{N}$ in σ_3, of an event point $\mathcal{E} \in \sigma_3$. We keep $\mathcal{N}$ fixed and let $\mathcal{V}_4$ shrink down to $\mathcal{N}$. Furthermore let (3.6) hold at $\mathcal{E}$, so that $n_\rho\, d\Sigma$ is proportional to $f_{,\rho}$. Then

$$[\mathcal{U}^{\rho\sigma}]\, f_{,\sigma} = 0 \quad , \quad f_{,r} = g N_r \quad \text{and} \quad c f_{,0} = -Vg = V g u_0 . \tag{4.7}$$

Thus (4.6) holds (in every frame).

5. KINEMATIC AND DYNAMIC CONSIDERATIONS ON CERTAIN EULERIAN AND LAGRANGIAN DISCONTINUITIES ACROSS SMALL SHOCK WAVES

The discontinuities across the material image σ_t^* of σ_3, as well as those across σ_3 itself, will be denoted by [ ]. Then there is a spatial vector B^ρ_*, related to σ_t^*, for which

$$[\alpha^\rho_L] \stackrel{!}{=} B^\rho_* \mathcal{N}^*_L \quad , \quad c\,[u^\rho] = -V_* B^\rho_* \qquad (B^\rho_* u_\rho = 0 \;\; , \;\; [u^\rho] = [u^\rho]\, \overset{\perp}{g}{}^\rho_\sigma) . \tag{5.1}$$

Indeed $(5.1)_{1,2}$ obviously hold for some B^ρ_*; and since $u^\alpha u_\alpha = -1$, by $(5.1)_2$, $(5.1)_3$ holds; hence $(5.1)_4$ also does. We also have

$$V\,[\mathcal{D}] = \mathcal{D} V_* B^\rho_* N_\rho \quad , \quad V\,[k] = ck\,[u^\rho]\, N_\rho \qquad (V_* \mathcal{D} N_\rho = V \gamma^L_\rho \mathcal{N}^*_L) . \tag{5.2}$$

Indeed by $(2.7)_3$ and $(3.4)_1$ $g_* \mathcal{N}^*_L \gamma^L_\rho = g \mathcal{D} N_\rho$, which by $(3.4)_2$ yields $(5.2)_3$. By $(2.7)_{1,2}$, $(2.6)_3$, and (3.6), the first of the relations

$$\mathcal{D} = \det \| \alpha^r_L \| \quad , \quad [\mathcal{D}] = \gamma^L_\rho [\alpha^\rho_L] = \gamma^L_\rho B^\rho_* \mathcal{N}^*_L = \frac{V_*}{V} \mathcal{D} B^\rho_* N_\rho \tag{5.2'}$$

holds. By $(2.7)_3$, $(5.1)_1$, and $(5.2)_3$ it yields $(5.2')_{2\text{-}4}$; hence $(5.2)_1$. By $(2.3)_2$ and $(2.7)_1$ $\mathcal{D} k = k^*$ (and k^* is supposed to be continuous). Then by $(5.2')_{2\text{-}4}$ and $(5.1)_2$, $V \mathcal{D}\,[k] = -V\,[\mathcal{D}]\, k = -V_* k^* B^\rho_* N_\rho = c k^* [u^\rho]\, N_\rho$; hence $(5.2)_2$.

q.e.d.

Since $[\overset{1}{g}{}^{\rho\sigma}] = 2\, u^{(\rho}\, [\dot{u}^{\sigma)}]$, by $(3.2)_3$ $g\,[g] = f_{/\rho}\, u^{\rho} f_{/\sigma}\, [u^{\sigma}]$, so that by $(3.2)_{2,1}$ and $(5.1)_4$

$$(5.3) \qquad [g] = -\frac{V}{c}\, f_{/\sigma}\, [u^{\sigma}] = -\frac{V}{c}\, g N_{\sigma}\, [u^{\sigma}]\,.$$

By $(3.2)_{2,1}$ and $(5.1)_4$ again, $[g]\,V + g\,[V] = -c f_{/\rho}\, [u^{\rho}] = -c g N_{\rho}\, [u^{\rho}]$, which by (5.3) yields

$$(5.4) \qquad [V] = -\left(1 - \frac{V^2}{c^2}\right) [c u^{\rho}\, N_{\rho}]\,.$$

Remark that this relativistic kinematic relation between $[V]$ and $[c u^{\rho}\, N_{\rho}]$ coincides with its classical correspondent (cf. [5, (185.6), p. 513]) only for $V = 0$.

Now remark that $(4.2)_3$ and the relations $X^{[\rho\sigma]} = 0 = X^{\rho\sigma}\, u_{\sigma}$ yield

$$(5.5) \qquad [q^{\rho}]\, u_{\rho} = -q^{\rho}\, [u_{\rho}] \quad , \quad [X^{\rho\sigma}] = [X^{\sigma\rho}] \quad , \quad [X_{\rho\sigma}]\, u^{\sigma} = -X_{\rho\sigma}\, [u^{\sigma}]\,;$$

hence by $(5.1)_4$, (3.6) *implies*

$$(5.6) \qquad [q^{0}] = q^{s}\, [u_s] \quad , \quad [X^{\rho 0}] = X^{\rho s}\, [u_s] \quad , \quad [X^{00}] = 0 = [u^{0}]\,.$$

Now let us choose (x) in such a way that, besides (3.6), we have $N_{\rho} = \delta^{3}_{\rho}$. By $(4.2)_2$ and $(3.6)_{4,2}$ $(u_0 = -1\,,\, -[\mathcal{U}^{\rho\sigma}]\, u_{\sigma} = [\mathcal{U}^{\rho 0}])$ and (5.6), *for such a choice of* (x) *the discontinuity relation* (4.6) *becomes*

$$(5.7) \qquad \begin{aligned} &\rho u^{\rho}\, [u^{3}] + [X^{\rho 3}] + 2\, [u^{(\rho}]\, q^{3)} + u^{\rho}\, [q^{3}] = \\ &= ([\rho]\, u^{\rho} + \rho\, [u^{\rho}] + X^{\rho s}\, [u_s] + u^{\rho}\, q^{s}\, [u_s] + [q^{\rho}])\, \frac{V}{c}\,. \end{aligned}$$

By $(3.6)_4$ and (5.6), for $\rho = r$ and $\rho = 0$ this simplifies into

$$(5.8) \qquad \left\{ \begin{aligned} &[X^{r3}] + 2\, q^{(3}\, [u^{r)}] = (\rho\, [u^{r}] + X^{rs}\, [u_s] + [q^{r}])\, \frac{V}{c}\,, \\ &\rho\, [u^{3}] + X^{3s}\, [u_s] + [q^{3}] = ([\rho] + 2\, q^{s}\, [u_s])\, \frac{V}{c} \end{aligned} \right.$$

respectively. From $(5.8)_1$ for $r = 3$ and from $(5.8)_2$ multiplied by Vc^{-1} we obtain

$$(5.9) \qquad [X^{33}] + 2\, q^{3}\, [u_3] = ([\rho] + 2\, q^{s}\, [u_s])\, \frac{V^2}{c^2} = [\rho]\, \frac{V^2}{c^2} + \boxed{4}$$

where $\boxed{r}$ means a term of the same order as c^{-r}, so that $q^{s}\, [u_s] = \boxed{2}$.

Remark that *if the heat flux is (spatially) orthogonal to the wave* σ_t, $(5.9)_1$ *simplifies into*

$$(5.9') \qquad [X^{33}] = [\rho]\, \frac{V^2}{c^2} - 2\left(1 - \frac{V^2}{c^2}\right) q^{3}\, [u_3] \qquad (q^{\rho} \parallel N^{\rho})\,.$$

Incidentally by $(5.2)_2$ and (2.3), *for* $N_\rho = \delta_\rho^3$

$$\frac{V}{c}[k] = k[u^3] \quad , \quad [\rho] = (c^2 + w)[k] + k[w] = \frac{c}{V}\rho[u^3] + k[w], \tag{5.10}$$

so that $(5.8)_2$ yields, *under condition* (3.6),

$$[q^3] = (k[w] + 2q^s[u_s])\frac{V}{c} - X^{3s}[u_s] \qquad (N_\rho = \delta_\rho^3). \tag{5.11}$$

Now let $\mathscr{C}$ be a non-viscous fluid $(X^{rs} = p\delta^{rs})$. *For it* $(5.8)_1$ *with* $r = 1, 2$ *and* $(5.9)_1$ *are equivalent*—by (5.10)—*to*

$$\begin{cases} \left(V\frac{kc^2 + kw + p}{c} - q^3\right)[u^h] = q^h[u^3] - \frac{V}{c}[q^h], & (h = 1, 2), \\ [p] = -\frac{2q^3}{ck}V[k] + \left\{\left(1 + \frac{w}{c^2}\right)[k] + \frac{k}{c^2}[w] + \frac{2}{c^2}q^s[u_s]\right\}V^2. \end{cases} \tag{5.12}$$

Incidentally remark that for $\mathscr{C}$ the classical analogue of $[cu_h]$—given by $(5.8)_1$ with $1/c = 0$, whence $q^r \equiv 0$—vanishes for $h = 1, 2$; (consequently) $[cu_h] = \boxed{2}$ by $(5.12)_1$ $(h = 1, 2)$. Then by $(5.10)_1$ *for non-viscous fluids and for* q^α *arbitrary* $(5.9)_1$ *(and* $(5.12)_2$*) become*

$$[p] = [\rho]\frac{V^2}{c^2} - 2\left(1 - \frac{V^2}{c^2}\right)q^3[u_3] + \boxed{6} = \tag{5.13}$$

$$= \left\{\left(1 + \frac{w}{c^2}\right)V^2 - \left(1 - \frac{V^2}{c^2}\right)\frac{2q^3}{ck}V\right\}[k] + \frac{k}{c^2}V^2[w] + \boxed{6}$$

where, for $q^\rho \parallel N^\rho$, $\boxed{6} = 0$ *rigorously.*

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