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Convergence of Fourier coefficients' series for vector valued functions

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Analisi matematica. — *Convergence of Fourier coefficients' series for vector valued functions.* Nota (*) di A. M. KANDIL, presentata dal Socio G. SANSONE.

RIASSUNTO. — Questa Nota è dedicata alla convergenza delle serie formate con i coefficienti di Fourier di funzioni periodiche, a valori vettoriali, mediante il modulo di continuità. Si prova che per tali funzioni, con valori in uno spazio di Hilbert, le serie sono assolutamente convergenti. D'altra parte quando i valori sono in uno spazio di Banach sequenzialmente debolmente completo la serie dei coefficienti di Fourier è fortemente incondizionatamente convergente. Viene dato un esempio.

1. INTRODUCTION.

This article is one of a series of studies carried out by the author on Fourier series for vector valued functions with values in a Banach space, and of strong bounded variation. Here the convergence of the Fourier coefficients is studied via the modulus of continuity.

We denote by X a Banach space with the norm $\|\cdot\|$. We study a periodic X-valued function $x(t)$ with the real variable t and of period 2π. The Fourier coefficients of $x(t)$ are defined as

$$C_k = \frac{1}{2\pi}\int_0^{2\pi} x(t)\exp(-ikt)\,dt, \qquad k = 0, 1, 2, \cdots.$$

On analogy to real valued functions, we define here the modulus of continuity $\omega(x, h)$ for $x(t)$ by the following:

$$\omega(x, h) = \sup_{|t_1 - t_2| \le h} \|x(t_1) - x(t_2)\|.$$

The integrated modulus of continuity is

$$\omega_p(x, h) = \sup_{|\eta| \le h}\left(\int_0^{2\pi} \|x(t+\eta) - x(t)\|^p\,dt\right)^{1/p}.$$

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We have to clarify the relations between the space values of a vector valued function and the convergence of its Fourier coefficients' series. The function is of strong bounded variation and satisfies the condition:

$$\sum_{n=1}^{\infty} \frac{\sqrt{\omega\left(x\,,\frac{1}{n}\right)}}{n} < \infty + \infty .$$

2. Theorem I

If:

(1) *$x(t)$ is a periodic vector valued function of the real variable t, with period 2π, and with values in a Hilbert space* H.

(2) *$x(t)$ is*: (i) *Strongly continuous and* (ii) *of strong bounded variation on* $[0\,,2\pi]$.

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{\omega\left(x\,,\frac{1}{n}\right)}}{n} < +\infty .$

Then

$$\sum_{k=-\infty}^{\infty} \|C_k\| < +\infty .$$

Proof of Theorem I

$x(t)$ is strongly continuous and of strong bounded variation on $[0\,,2\pi]$, hence we have [3].

$$x(t+h)-x(t) = \sum_{k=-\infty}^{\infty} C_k\,(e^{ikh}-1)\,e^{ikt}.$$

Applying Parseval's identity [7], one gets

$$\frac{1}{2\pi}\int_0^{2\pi} \|x(t+h)-x(t)\|^2\,dt = \sum_{k=-\infty}^{\infty} |e^{ikh}-1|^2\,\|C_k\|^2 = 4\sum_{k=-\infty}^{+\infty} \|C_k\|^2\cdot\sin^2\frac{kh}{2}\,.$$

So,

$$4\sum_{k=-\infty}^{\infty} \|C_k\|^2 \sin^2\frac{kh}{2} = \frac{1}{2\pi}\,\omega_2^2(x\,,h)\,.$$

Let $h = \dfrac{\pi}{2^{p+1}}$; the above equation becomes:

$$4\sum_{k=2^{p-1}}^{2^p} \|C_k\|^2 \sin^2\frac{k\pi}{2^{p+1}} \leqq \frac{1}{2\pi}\,\omega_2^2(x\,,2^{-p-1}\pi)\,.$$

and thus

$$\sum_{k=2^{p-1}}^{2^p} \|C_k\|^2 \leqq \frac{1}{4\pi} \omega_2^2 (x , 2^{-p-1} \pi) .$$

By Cauchy's inequality

$$\sum_{k=2^{p-1}}^{2^p} \|C_k\|^2 \leqq \left(\sum_{k=2^{p-1}}^{2^p} \|C_k\|^2 2^{p-1} \right)^{\frac{1}{2}}$$

$$< \frac{2^{p/2}}{2\sqrt{2\pi}} \omega_2 (x , 2^{-p-1}) .$$

Then if,

$$\text{(1)} \qquad \sum_{n=1}^{\infty} \frac{\omega_2\left(x , \frac{1}{n}\right)}{\sqrt{n}} < +\infty$$

i.e.,

$$\sum_{p=1}^{\infty} 2^{p/2} \omega_2 (x , 2^{-p}) < +\infty$$

one gets

$$\sum_{k=0}^{\infty} \|C_k\| < +\infty .$$

With the same procedure for negative values of k, we deduce that if (1) is valid then

$$\sum_{k=-\infty}^{+\infty} \|C_k\| < +\infty .$$

We deduce the relation between $\omega_2 (x , 1/n)$ and $\omega (x , 1/n)$ when $x(t)$ is a vector valued function of strong bounded variation.

Let, $0 \leq \eta \leq h$

$$\int_0^{2\pi} \|x(t+\eta) - x(t-\eta)\|^2 \, dt < \max_{0 \leq t \leq 2\pi} \|x(t+\eta) - x(t-\eta)\| \cdot$$

$$\cdot \int_0^{2\pi} \|x(t+\eta) - x(t-\eta)\| \, dt .$$

So we have,

$$\omega_2^2 (x , h) \leqq 4 \, \omega (x , h) , \omega_1 (x , h) .$$

Since $x(t)$ is of strong bounded variation, we get

$$\omega_2(x, h) < A\sqrt{h\omega(x, h)} \qquad \text{(A is a constant).}$$

So if

$$\sum_{n=1}^{\infty} \frac{\sqrt{\omega\left(x, \frac{1}{n}\right)}}{n} < +\infty .$$

we get

$$\sum_{n=1}^{\infty} \frac{\omega_2\left(x, \frac{1}{n}\right)}{\sqrt{n}} < +\infty$$

and this completes the proof.

3. Theorem II

If

(1) $x(t)$ *is a periodic vector valued function of the real variable* t, *with period* 2π *and with values in a weakly sequentially complete Banach space.*

(2) $x(t)$ *is*: (i) *strongly continuous, and* (ii) *of strong bounded variation on* $[0, 2\pi]$

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{\omega\left(x, \frac{1}{n}\right)}}{n} < +\infty .$

Then the series $\sum_{n=-\infty}^{+\infty} C_n$ is strongly unconditionally convergent.

Proof of Theorem II

Let X^* be the conjugate space of X. Consider the numerical function $x^* x(t)$ where x^* is a linear functional in X^*. The Fourier coefficients of $x^* x(t)$ are $x^* C_k$ $(k = 0, \pm 1, \pm 2, \cdots)$. Apply Perseval's identity to $x^* x(t)$; we get

$$4\sum_{-\infty}^{+\infty} |x^* C_k|^2 \sin^2 \frac{kh}{2} = \frac{1}{2\pi}\int_0^{2\pi} |x^*(x(t+h) - x(t))|^2 \, dt$$

$$< \frac{1}{2\pi}\int_0^{2\pi} \|x(t+h) - x(t)\|^2 \, dt .$$

Repeating the steps of the previous theorem, one get

$$\sum_{-\infty}^{+\infty} |x^* C_k| < +\infty .$$

So, any subsequence of the partial sums of the series $\sum_{k=-\infty}^{+\infty} C_k$ is weakly convergent. Using the given condition that the space of the values is a weakly sequentially complete Banach space, we deduce that the series $\sum_{k=-\infty}^{+\infty} C_k$ is weakly unconditional convergent.

Apply the fact that in Banach space, each weakly unconditional convergent series is strongly unconditional convergent [2]. Hence the theorem is proved.

4. EXAMPLE

In this section we introduce a L (0 , 2 π) valued function, defined on [0 , 2 π] and satisfing *all the requirements of theorem II*. We show that the Fourier coefficients' series is not absolutely convergent.

Take $x(t)$, defined as follows

$$x(t) = \begin{cases} 1 & 0 \leq \sigma \leq t \\ 0 & t < \sigma \leq 2\pi . \end{cases}$$

It is clear that

$$\|x(t+\eta) - x(t)\| = \int_t^{t+\eta} dt = \eta .$$

So, $x(t)$ is strongly continuous and of strong bounded variation on [0 , 2 π]. Also,

$$\omega\left(x , \frac{1}{n}\right) = \sup_{|h| \leq 1/n} \|x(t+h) - x(t)\| = \frac{1}{n} \cdot$$

So, $\sum_{n=1}^{\infty} \frac{\sqrt{\omega\left(x , \frac{1}{n}\right)}}{n} < +\infty .$

It is easy to calculate its Fourier series coefficients with the result:

$$C_0 = \left(1 - \frac{\sigma}{2\pi}\right) \quad , \quad C_k : \frac{i}{2\pi k} \{(1 - \cos k\sigma) + i \sin k\sigma\}$$

where: $\|C_0\| = \pi$

$$\|C_k\| = \int_0^{2\pi} |C_k| \, d\sigma = \int_0^{2\pi} \left| \frac{\sin \frac{k}{2} \sigma}{k} \right| d\sigma = \frac{4}{\pi |k|}$$

and hence the Fourier coefficients' series is not absolutely convergent.

This means that the conditions of Theorem II are insufficient for the absolute convergence of the Fourier coefficients' series.

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