Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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On the asymptotic behaviour of Boltzmann's h function in the Kinetic Theory of Gases

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **67** (1979), n.3-4, p. 248–251.

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<http://www.bdim.eu/item?id=RLINA_1979_8_67_3-4_248_0>

Meccanica. — *On the asymptotic behaviour of Boltzmann's* h *function in the Kinetic Theory of Gases* (*). Nota (**) di MARIO PITTERI, presentata dal Socio straniero C. TRUESDELL.

RIASSUNTO. — In questa Nota si mostra, mediante un esempio, che un'affermazione tradizionale sul comportamento asintotico della funzione h di Boltzmann non è in generale corretta.

§ 1. INTRODUCTION

In Appendix B to Chapter XVII of their book [5], Truesdell and Muncaster mention an analysis of mine regarding the asymptotic behavior of Boltzmann's h function in the Kinetic Theory of Gases. In this Note I present that analysis. I presume the reader to be already familiar with the content of the book [5], in particular with Chapters VIII, IX and Appendix B to Chapter XVII, and I use the same notations.

In § 2 I specify a suitable subfamily of the solutions of the Maxwell-Boltzmann equations discovered by Muncaster [4], and show that for each member of that subfamily Boltzmann's function h exists for all positive times and decreases strictly as time increases. According to a traditional claim, presented for instance by Chapman and Cowling [1, § 4.1], the function h specified above should tend asymptotically to the value corresponding to a Maxwellian density, and the gas should approach equilibrium. I consider the difference between h and its counterpart h_M for the Maxwellian density F_M that has the same principal moment as F. I show that, for every molecular density in the aforementioned subfamily, the difference $h - h_M$ decreases strictly but does not tend to zero as time goes to infinity. Hence the first part of the traditional claim above is not always true. The same can be said of the second part of the claim, as Truesdell and Muncaster [5] show, for instance, in Appendix B to Chapter XVII.

§ 2. ASYMPTOTIC BEHAVIOR OF h FOR A FAMILY OF EXACT SOLUTIONS OF THE MAXWELL-BOLTZMANN EQUATION

Let us call $F_M(\rho, \boldsymbol{u}, \varepsilon)$ the locally Maxwellian density that delivers the same values of ρ, $\boldsymbol{u}$ and ε as does a given molecular density F. The density $F_M(\rho, \boldsymbol{u}, \varepsilon)$ has the form

$$F_M(\rho, \boldsymbol{u}, \varepsilon) = a\, e^{-(bc^2/2)} \quad , \quad c = \| \boldsymbol{v} - \boldsymbol{u} \| , \tag{2.1}$$

(*) Lavoro eseguito in base ad accordi tra la U. S. National Science Foundation e la Johns Hopkins University, ove l'autore ha inoltre usufruito di una borsa di studio del C.N.R.

(**) Pervenuta all'Accademia l'8 ottobre 1979.

where

$$a = \frac{\rho}{\mathfrak{m}\left(\frac{4}{3}\pi\varepsilon\right)^{3/2}} \quad \text{and} \quad b = \frac{3}{4\,\varepsilon}\,. \tag{2.2}$$

In Chapters VIII and XI, Truesdell and Muncaster [5] show that the corresponding functions h and h_M satisfy the conditions

$$h - h_M \geqq 0 \tag{2.3}$$

and

$$\rho\,(h - h_M) \leqq \frac{1}{\frac{2}{3}\varepsilon}\left(-\mathbf{P}\cdot\mathbf{E} - \frac{1}{\varepsilon}\,\boldsymbol{q}\cdot\operatorname{grad}\varepsilon\right) - \operatorname{div}\left(\boldsymbol{s} + \frac{1}{\frac{2}{3}\boldsymbol{q}}\right), \tag{2.4}$$

provided they exist and are smooth enough. Moreover, equality holds in (2.3) and (2.4) if and only if $F = F_M(\rho\,,\boldsymbol{u}\,,\varepsilon)$, in which case both sides of (2.4) vanish.

Following Muncaster [4], let us consider the family of functions

$$F = F_M(\rho\,,\boldsymbol{u}\,,\xi)\,(A + B\,\|\,\boldsymbol{v} - \boldsymbol{u}\,\|^2)\,, \tag{2.5}$$

where

$$A = 1 - \frac{3}{2}\left(\frac{\varepsilon}{\xi} - 1\right) \quad \text{and} \quad 2\,\xi B = 1 - A\,. \tag{2.6}$$

Moreover $F_M(\rho\,,\boldsymbol{u}\,,\xi)$ is the function we obtain when we replace ε in $F_M(\rho\,,\boldsymbol{u}\,,\varepsilon)$ by the positive real variable ξ, subject to the condition

$$1 \leqq \frac{\varepsilon}{\xi} \leqq \frac{5}{3}\,, \tag{2.7}$$

which implies that $F \geqq 0$. Therefore the function F in (2.5) is acceptable as a molecular density for such values of ε/ξ. In addition, as the notation suggests, it is possible to show that the molecular density defined by (2.5) and (2.6) has ρ , $\boldsymbol{u}$, and ε as its fields of density, gross velocity and energetic. The equations of balance of mass, momentum and energy which the kinetic gas satisfies in undergoing a homo-energetic dilatation subject to vanishing body force have the particular solution

$$\begin{aligned} \rho &= \rho(0)\,(1 + t/T)^{-3}, \\ \boldsymbol{u} &= (t + T)^{-1}\,\boldsymbol{r}, \\ \varepsilon &= \varepsilon(0)\,(1 + t/T)^{-2}. \end{aligned} \tag{2.8}$$

For these particular fields ρ , $\boldsymbol{u}$, and ε, Muncaster [4] shows that (2.5) is a solution of the Maxwell-Boltzmann equation for a gas of Maxwellian molecules. Moreover

$$\frac{\xi}{\varepsilon} = 1 + \left(\frac{\xi(0)}{\varepsilon(0)} - 1\right)\exp\left[\frac{T^3}{6\,\tau(0)}\left(\frac{1}{(t+T)^2} - \frac{1}{T^2}\right)\right], \tag{2.9}$$

where $\tau(0)$ is a positive number and T is a parameter. In particular, if we choose $T > 0$, the solution above exists for all $t \geqq 0$. For every choice of ε and ξ that satisfies (2.7), the integrals that define h and $\boldsymbol{s}$ converge, and $\boldsymbol{s}$ vanishes because it is proportional to the integral of $\boldsymbol{c}$ times a function of c^2. Moreover, if we denote by $\tilde{\mathrm{F}}$ and $\tilde{a}$ the first factor in (2.5) and its coefficient a respectively, we have that

$$(2.10) \qquad h - h_{\mathrm{M}} = \frac{1}{n}\left(\int \mathrm{F}\log\mathrm{F} - \int \mathrm{F_M}\log\mathrm{F_M}\right)$$

$$= \frac{\mathrm{A}}{n}\int \tilde{\mathrm{F}}\log\tilde{\mathrm{F}} + \frac{\mathrm{B}}{n}\int \tilde{\mathrm{F}}\,c^2\log\tilde{\mathrm{F}} - \frac{1}{n}\int \mathrm{F_M}\log\mathrm{F_M} +$$

$$+ \frac{1}{n}\int \tilde{\mathrm{F}}\,(\mathrm{A} + \mathrm{B}c^2)\log(\mathrm{A} + \mathrm{B}c^2)\,.$$

We can evaluate the first three integrals in (2.10) by use of the formula

$$(2.11) \qquad \int_0^\infty c^{2n}\,e^{-bc^2}\,\mathrm{d}c = \frac{1\cdot 3\cdot\;\cdots\;\cdot(2n-1)}{2^{n+1}\,b^n}\sqrt{\frac{\pi}{b}}$$

and so show that

$$(2.12) \qquad h - h_{\mathrm{M}} = \mathrm{A}\left(\log\tilde{a} - \frac{3}{2}\right) + 2\,\mathrm{B}\xi\log\tilde{a} - 5\,\mathrm{B}\xi - \log a +$$

$$+ \frac{3}{2} + \frac{1}{n}\int \tilde{\mathrm{F}}\,(\mathrm{A} + \mathrm{B}c^2)\log(\mathrm{A} + \mathrm{B}c^2)$$

$$= \frac{3}{2}\log\frac{\varepsilon}{\xi} + \mathrm{A} - 1 + \frac{1}{n}\int \tilde{\mathrm{F}}\,(\mathrm{A} + \mathrm{B}c^2)\log(\mathrm{A} + \mathrm{B}c^2)\,.$$

Integrating the last term in $(2.12)_2$ over the angular variables and then substituting $y = c^2/2\,\xi$ in the integral that remains, we transform $(2.12)_2$ into

$$(2.13) \qquad h - h_{\mathrm{M}} = \frac{3}{2}\log\frac{\varepsilon}{\xi} + \mathrm{A} - 1 + \frac{3}{2}\sqrt{\frac{6}{\pi}}\int_0^\infty e^{-(3/2)y}\,\cdot$$

$$\sqrt{y}\,[\mathrm{A} + (1 - \mathrm{A})\,y]\log[\mathrm{A} + (1 - \mathrm{A})\,y]\,\mathrm{d}y\,.$$

Therefore we conclude that $h - h_{\mathrm{M}}$ is a continuously differentiable function of ε/ξ. In (2.9) ε/ξ is a continously differentiable function of t on the whole half line $[0, +\infty)$ if $T > 0$; hence so is also $h - h_{\mathrm{M}}$. Muncaster [4] shows that $\mathbf{P} = 0 = \boldsymbol{q}$ for all the functions (2.5) restricted by (2.7). Therefore the inequalities (2.3) and (2.4) reduce to

$$(2.14) \qquad h - h_{\mathrm{M}} \geqq 0\,, \qquad (h - h_{\mathrm{M}})^{\boldsymbol{\cdot}} \leqq 0$$

for such functions, and the equalities hold if and only if $\mathrm{F} = \mathrm{F_M}(\rho, \boldsymbol{u}, \varepsilon)$.

In the case $\xi/\varepsilon = 3/5$, it is possible to evaluate $h - h_M$ explicity:

$$(2.15) \qquad h - h_M = \frac{3}{2}\log\frac{5}{3} - 1 + \frac{3}{2}\sqrt{\frac{6}{\pi}}\int_0^\infty e^{-(3/2)y} y^{3/2}\log y\,dy$$

$$= \frac{3}{2}\log\frac{5}{3} - 1 + \frac{3}{2}\sqrt{\frac{6}{\pi}}\left(\Gamma\left(\frac{5}{2}\right)\left(\frac{2}{3}\right)^{5/2}\left[\psi\left(\frac{5}{2}\right) - \log\frac{3}{2}\right]\right)$$

$$= \frac{3}{2}\log\frac{5}{3} + \frac{5}{3} - \gamma - \log 6\,.$$

The parenthesis in $(2.15)_2$ is the value of the integral in $(2.15)_1$, which is a known Laplace transform (1), γ is Euler's constant and $\Gamma(x)$, $\psi(x)$ are the classical functions that the reader may find, for instance, in Erdélyi [2, pp. 1 and 15]. A quick numerical evaluation shows that the right hand side in $(2.15)_3$ is positive.

Because $h - h_M$ is continuous as a function of ξ/ε, and because this ratio depends on t continuously by (2.9), we can fix independently both the initial values $\frac{\xi(0)}{\varepsilon(0)}$ and $\frac{T}{6\tau(0)}$ in such a way that

$$1 + \left(\frac{\xi(0)}{\varepsilon(0)} - 1\right)\exp\left(-\frac{T}{6\tau(0)}\right)$$

shall belong to a neighborhood of $\xi/\varepsilon = 3/5$ in which $h - h_M$ is a positive function of ξ/ε. With any such choice, $h - h_M$ decreases as t tends to infinity, but its asymptotic value is not zero.

REFERENCES

[1] S. CHAPMAN and T. G. COWLING (1939) - *The Mathematical Theory of Non-Uniform Gases*, Cambridge University Press.

[2] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER and F. TRICOMI (1954) - *Tables of Integral Transforms*, Vol. I, McGraw-Hill Book Company, New York etc.

[3] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER and F. TRICOMI (1953) - *Higher Trascendental Functions*, Vol. I, McGraw-Hill Book Company, New York etc.

[4] R. G. MUNCASTER (1979) - *On Generating Exact Solutions of the Maxwell-Boltzmann Equation*, «Archive for Rational Mechanics and Analysis», 70, 79-90.

[5] C. TRUESDELL and R. G. MUNCASTER (1979) - *Fundamentals of Maxwell's Kinetic Theory of a Simple Monatomic Gas, treated as a branch of Rational Mechanics*, «Pure and Applied Mathematics», Academic Press, New York.

(1) See for instance Erdélyi [3, p. 148].