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On a metric of E. Kähler

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Geometria differenziale. — *On a metric of E. Kähler.* Nota (*) di FROIM MARCUS, presentata dal Socio G. SANSONE.

RIASSUNTO. — Si dimostra che le metriche studiate da E. Kähler in un suo lavoro come esempio di metriche che appartengono alla teoria delle funzioni automorfe sono semplicemente metriche simili a quelle di G. Fubini.

1. In the paper, *Über eine bemerkenswerte Hermitesche Metrik*, E. Kähler [1], showed that in a study of the invariants of a real $2n$–dimensional Hermitian metric

$$ds^2 = \Sigma g_{i\bar{k}}\, dx_i\, d\bar{x}_k\,, \tag{1.1}$$

in the context of the pseudoconformal transformation

$$x'_i = \phi_i(x_1 x_2, \cdots, x_n) \quad ; \quad \bar{x}'_i = \bar{\phi}_i(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \quad (i = 1, 2, \cdots, n) \tag{1.2}$$

it is useful to consider the exterior form

$$\omega = \Sigma g_{i\bar{k}}\, d(x_i, \bar{x}_k)\,, \tag{1.3}$$

where $\bar{x}$ are the complex conjugates of x. Kähler, observed that $\omega' = 0$ is a noteworthy exception: it turns out that in this case, the metric

$$ds^2 = \Sigma \frac{\partial^2 U}{\partial x_i\, \partial \bar{x}_k}\, dx_i\, d\bar{x}_k\,, \tag{1.4}$$

derives from a potential which is evidently an invariant equivalent to the condition $\omega' = 0$. This type, as Kähler observes, comprises some metrics in the theory of automorphic functions.

If

$$x'_i = \frac{L_i(x)}{L_0(x)} = \frac{\alpha_{i0} + \alpha_{i1} x_1 + \cdots + \alpha_{in} x_n}{\alpha_{00} + \alpha_{01} x_1 + \cdots + \alpha_{0n} x_n}\,, \qquad (i = 1, 2, \cdots, n) \tag{1.4'}$$

are projective transformations which transform the hypersphere

$$1 - x_1 \bar{x}_1 - x_2 \bar{x}_2 - \cdots - x_n \bar{x}_n = 0\,, \tag{1.5}$$

(*) Pervenuta all'Accademia il 7 settembre 1979.

into itself, then, because

$$\left(1 - \sum_v x'_v \overline{x}'_v\right) = \left(1 - \sum_v x_v \overline{x}_v\right) (\mathrm{L}_0(x) \cdot \overline{\mathrm{L}}_0(\overline{x}))^{-1}, \tag{1.6}$$

the metric

$$ds^2 = \Sigma \frac{\partial^2 \mathrm{U}}{\partial x_i \, \partial \overline{x}_k} \, dx_i \, d\overline{x}_k , \tag{1.6'}$$

where

$$\mathrm{U} = k \log (1 - \Sigma x_v \overline{x}_v) , \qquad (k \text{ a constant}) \tag{1.7}$$

is invariant under the group of hyperfuchsian transformations (1.4').

It should be noted that it is unclear to the reader how Kähler arrived at his metric. *Likewise, one may find it very strange that he makes no mention at all of the fundamental papers by Guido Fubini on the quadratic Hermitian forms and their metrics.*

The aim of our paper is to show that if (1.7) *holds, the metric presented by Kähler is none other than that of Fubini multiplied by a constant.*

2. To prove our point, we refer first of all to Fubini's paper [2] (G. Fubini, Opere Scelte, Edizione Cremonese, Roma 1957, Vol. I (1), pp. 307–311). In it, Fubini considers a Hermitian form reduced to the type

$$x_1 \overset{0}{x}_1 + x_2 \overset{0}{x}_2 + \cdots + x_{n-1} \overset{0}{x}_{n-1} - x_n \overset{0}{x}_n , \tag{2.1}$$

in n variables x_i, of which $\overset{0}{x}_i$ are the complex conjugates. He considers the group of transformations

$$x'_i = \sum_k a_{ik} x_k , \tag{2.2}$$

with $| a_{ik} | = 1$, $i = 1, 2, \cdots, n$, which transforms the given form into itself.

Then the transformation

$$u'_i = \frac{a_{i1} u_1 + a_{i2} u_2 + \cdots + a_{i,n-1} u_{n-1} + a_{in}}{a_{n1} u_1 + a_{n2} u_2 + \cdots + a_{n,n-1} u_{n-1} + a_{n,n}} , \tag{2.3}$$

of the variables $u_1, u_2, \cdots, u_{n-1}$, where

$$u_i = \frac{x_i}{x_n} ,$$

forms a group which transforms the hypervariety (in the sense of C. Segre)

$$\sum_1^{n-1} u_i \overset{0}{u}_i - 1 = 0 , \tag{2.4}$$

(1) The three volumes are designated in the sequel as [F_1], [F_2] and [F_3].

into itself. Putting $u_i = u_i' + iu_i''$, $\overset{0}{u}_i = u_i - iu_i''$, $(i = 1, 2, \cdots, n-1)$, (2.4) is $(i = \sqrt{-1})$ the hypersphere

$$(2.5) \qquad (u_1')^2 + (u_1'')^2 + (u_2')^2 + (u_2'')^2 + \cdots - 1 = 0 .$$

Fubini now proceeds to prove that the pair of points $(u_i' \, u_i'')$ *and* $(\bar{u}_i', \bar{u}_i'')$ have a symmetrical invariant in the pair connected with all transformations under (2.3) which satisfy

$$S' = \frac{S}{\operatorname{mod}^2 (a_{n1} u_1 + a_{n2} u_2 + \cdots + a_{n,n-1} u_{n-1} + a_{nn})},$$

where S is the left hand side of (2.5), and S′ its transformation under (2.3). This invariant is given by

$$(\alpha) \qquad R_{u\bar{u}} = \frac{\left(\sum_1^{n-1} u_i \overset{0}{\bar{u}}_i - 1\right)\left(\sum_1^{n-1} \overset{0}{u}_i \bar{u}_i - 1\right)}{\left(\sum_1^{n-1} u_i \overset{0}{u}_i - 1\right)\left(\sum_1^{n-1} \bar{u}_i \overset{0}{\bar{u}}_i - 1\right)} - 1 .$$

Indeed, if $v_i, \bar{v}_i$ are the transformations of u_i and $\bar{u}_i$ under (2.3), then we have expressions of the form

$$(\delta_1) \qquad \sum_1^{n-1} v_i \overset{0}{v}_i - 1 = \frac{\sum_1^{n-1} u_i \overset{0}{u}_i - 1}{A \cdot B};$$

where

$$A = a_{n,1} u_1 + \cdots + a_{n,n-1} u_{n-1} + a_{nn} \quad ; \quad B = \overset{0}{a}_{n1} \overset{0}{u}_1 + \cdots + a_{n,n-1} \overset{0}{u}_{n-1} + \overset{0}{a}_{nn} .$$

$R_{u\bar{u}}$ is real; if it is infinitesimal, the points u and $\bar{u}$ are infinitesimally close. *Fubini calls* $\sqrt{R_{u\bar{u}}}$ *the pseudodistance of the points u, and* $\bar{u}$, *and* (2.3) *a pseudo–motion.*

Fubini shows that (α) can also be written out in the explicit form

$$(\alpha_1) \qquad R_{u\bar{u}} = \frac{\begin{pmatrix} u_1 \, u_2, \cdots, u_{n-1}, i \\ \bar{u}_1 \, \bar{u}_2, \cdots, \bar{u}_{n-1}, i \end{pmatrix} \begin{pmatrix} \overset{0}{\bar{u}}_1 \, \overset{0}{\bar{u}}_2, \cdots, \overset{0}{\bar{u}}_{n-1}, i \\ \overset{0}{u}_1 \, \overset{0}{u}_2, \cdots, \overset{0}{u}_{n-1}, i \end{pmatrix}}{\left(\sum_1^{n-1} u_i \overset{0}{u}_i - 1\right)\left(\sum_1^{n-1} \bar{u}_i \overset{0}{\bar{u}}_i - 1\right)}, \qquad (i = \sqrt{-1})$$

with multiplication effected row by row. It is assumed that the two points are inside (2.5).

3. In another paper [3], F_1 pp. 325–332, Fubini presents an analytical application of the projective groups which transform Hermitian form into itself. For simplicity, he considers the following case. *Let* $xx_0 + yy_0 - zz_0$

be an indefinite Hermitian form A, x_0, y_0, z_0 the complex conjugate variables of x, y, z, and

$$(3.1)\qquad \frac{x}{z} = u_1 = u_1' + iu_1'' \quad ; \quad \frac{y}{z} = u_2 = u_2' + iu_2'' ,$$

where u_1', u_2', u_1'', u_2'' are real variables. To every linear homogeneous transformation T of x, y, z, there corresponds (generally) a linear transformation T′ of the form (2.3) of the variables $u_1\, u_2$. Consider the transformation T, which leaves the form A and the corresponding transformation T′ unchanged. In the space R where u_1', u_1'', u_2', u_2'' are the coordinate variables, the transformations T′ constitute a continuous group which can be considered as a group of motions of a metric defined by the real definite linear element

$$(\beta)\qquad ds^2 = \frac{(1 - u_2 u_2^0)\, du_1\, du_1^0 + (1 - u_1 u_1^0)\, du_2\, du_2^0 + u_1 u_2^0\, du_2\, du_1^0 + u_2 u_1^0\, du_1\, du_2^0}{(1 - u_1 u_1^0 - u_2 u_2^0)^2}$$

where $u_1 = u_1' + iu_1''$, $u_2 = u_2' + iu_2''$, etc.

Fubini observed that this linear element is obtainable either directly or *from his formulas* (α_1) *for a pair of infinitesimally close points.*

It is seen from (β) that *the coefficients of* $du_i\, du_k^0$ *equal*

$$(\gamma)\qquad \frac{\partial^2 \log \left(\sum_1^2 1 - u_i u_i^0 \right)^{-1}}{\partial u_i\, \partial u_k^0} = \frac{\partial^2 \log (1 - u_1 u_1^0 - u_2 u_2^0)^{-1}}{\partial u_i\, \partial u_k^0} ,$$

and setting

$$(3.2)\qquad U = -\log \sum_1^2 (1 - u_i u_i^0) = -\log (1 - u_1 u_1^0 - u_2 u_2^0) ,$$

this is $k = -i$ in (1.7), becomes

$$(\beta_1)\qquad ds^2 = \Sigma \frac{\partial^2 U}{\partial u_i\, \partial u_k^0}\, du_i\, du_k^0 .$$

It can be proved that for a pair of infinitesimally close points $u, \bar{u}$ *in a* R_{n-1} *space, the denominator*

$$(3.3)\qquad \left(\sum_1^{n-1} u_i u_i^0 - 1 \right) \left(\sum_1^{n-1} \bar{u}_i \bar{u}_i^0 - 1 \right) ,$$

of (α_1) is

$$(3.4)\qquad \left(\sum_1^{n-1} u_i u_i^0 - 1 \right)^2 .$$

Then, the numerator od (α_1) *yields the coefficients of Fubini's corresponding metric; and comparison with* (1.6') *establishes that Kähler's metric is none other than Fubini's one, multiplied by a constant.*

q. e. d.

Still later (see F_2, p. 111), Fubini [4] showed that the projective transformations of an Hermitian form

$$x_1 \overset{0}{x}_1 + x_2 \overset{0}{x}_2 + \cdots + x_{n-1} \overset{0}{x}_{n-1} \pm x_n \overset{0}{x}_n , \tag{6}$$

into itself become, when expressed in terms of $u_i , \overset{0}{u}_i$, a group of motion in the real metric

$$ds^2 = \frac{\begin{pmatrix} u_1 , u_2 , \cdots , u_{n-1} , \sqrt{\pm 1} \\ du_1 , du_2 , \cdots , du_{n-1} , 0 \end{pmatrix} \begin{pmatrix} d\overset{0}{u}_1 , d\overset{0}{u}_2 , \cdots , d\overset{0}{u}_{n-1} , 0 \\ \overset{0}{u}_1 , \overset{0}{u}_2 , \cdots , \overset{0}{u}_{n-1} , \sqrt{\pm 1} \end{pmatrix}}{\left(\sum_1^{n-1} u_i \overset{0}{u}_i \pm 1 \right)^2} , \tag{β_2}$$

which we shall write under the following form:

$$\frac{\begin{vmatrix} \sum_1^{n-1} u_i \, d\overset{0}{u}_i & \sum_1^{n-1} u_i \overset{0}{u}_i \pm 1 \\ \sum_1^{n-1} du_i \, d\overset{0}{u}_1 & \sum_1^{n-1} \overset{0}{u}_i \, du_i \end{vmatrix}}{\left(\sum_1^{n-1} u_i \overset{0}{u}_i \pm 1 \right)^2} , \tag{β_3}$$

which is more expressive.

It is seen that (3.4) *is of the same form as the denominator of* (β_2) *which can thus be included under the form* (β_1).

It is necessary to remark that in this paper Fubini gives other very important results of discontinuous groups, the generalization of Poincaré's Zetafuchsian functions, etc.

Incidentally, in his paper « Chatacteristic classes of Hermitian manifolds ", Shüng–Shen–Chern [6] considered a class of Hermitian metrics, *called Hermitian metrics qithout torsion,* and noted that these are the metrics introduced by Kähler in [1].

It thus follows that *Fubini's metrics and those multiplied by a constant, are Hermitian metric without torsion.*

References

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