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Elliptic Equations Containing the Term $r\frac{\partial}{\partial r}$ in a Half Space

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Equazioni a derivate parziali. — *Elliptic Equations Containing the Term* $r\frac{\partial}{\partial r}$ *in a Half Space* (*). Nota (**) di ANTONIO BOVE (***), BRUNO FRANCHI (***) e ENRICO OBRECHT (****), presentata dal Corrisp. G. CIMMINO.

RIASSUNTO. — In questa Nota si considerano, in opportuni spazi con peso, problemi al contorno in un semispazio per operatori del tipo $A(x, \partial) + x \cdot \nabla + q^2$ dove $A(x, \partial)$ è un operatore ellittico e q è un parametro complesso.

1. In this paper we consider a boundary value problem in a half-space for a second order equation having as its principal part an elliptic operator and containing in its lower order terms a radial derivative whose modulus grows up as $|x| \to +\infty$.

More specifically, we study operators like

$$\Delta + x \cdot \nabla + q^2 \tag{1}$$

where q is a complex parameter, which are important in the study of the regularity of parabolic problems with mixed lateral conditions near a point in which the "separating surface" is characteristic (see [3]).

By a change of the unknown function, e.g. $u = \exp[\alpha |x|^2] V$, α a suitable real number, it is possible to reduce this operator to an operator elliptic in the pair of variables (x, ξ), which was studied in [4], [5].

But the spaces so obtained have a weight function exponentially increasing at infinity and so we get a result not as precise as desirable.

Complete proofs of the results announced here will be published elsewhere ([6]).

2. Let $q \in \mathbf{C}$, $\operatorname{Re} q^2 \leq 0$. The operator $A_q = q^2 + x \cdot \nabla$ generates in $L^2(\mathbf{R}^n)$ a C_0-semigroup of type $\operatorname{Re} q^2 - n/2$; hence, it is possible to define the fractional powers of $-A_q$.

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If m is a non negative integer, put

$$M_q^0(\mathbf{R}^n) = L^2(\mathbf{R}^n)\,,\ |||\,u\,|||_0 = \|\,u\,;L^2(\mathbf{R}^n)\,\|\,,$$

$$M_q^{2m+2}(\mathbf{R}^n) = \mathscr{D}_{M_q^{2m}}(1-\Delta)\cap\mathscr{D}_{M_q^{2m}}(-A_q)\,,$$

$$|||\,u\,|||_{2m+2} = (|||\,(1-\Delta)\,u\,|||_{2m}^2 + |||\,A_q\,u\,|||_{2m}^2 + |||\,u\,|||_{2m}^2)^{1/2}\,,$$

where $\mathscr{D}_{M_q^{2m}}(B)$ denotes the domain of B in $M_q^{2m}(\mathbf{R}^n)$.

Moreover, if $s \in\,]0,1[$, put

$$M_q^{2m+2s}(\mathbf{R}^n) = \mathscr{D}_{M_q^{2m}}((1-\Delta)^s)\cap\mathscr{D}_{M_q^{2m}}((-A_q)^s),$$

normed analogously to the even integer case.

This definition is correct, since $-A_q$ has fractional powers in $M_q^{2m}(\mathbf{R}^n)$, too.

Spaces $M_q^s(\mathbf{R}_+^n)$ are defined as quotient spaces.

It is of fundamental importance that our spaces have the interpolation property. To prove this fact, we need a non commutative interpolation theorem ([7]) as well as some results about non self-adjoint operators ([8], [9], [10]).

These spaces — as subsets of Sobolev spaces — have ordinary traces on the hyperplane $x_n = 0$.

Moreover, let $s > p + 1/2$ and denote by $\gamma_k u$ the *k-th* order trace of u; the map

$$u \rightarrow (\gamma_0\,u\,,\cdots,\gamma_p\,u)$$

is bounded from $M_q^s(\mathbf{R}^n)$ onto $\displaystyle\mathop{\times}_{j=0}^{p} M_q^{s-j-1/2}(\mathbf{R}^{n-1})$ and its norm is uniformly bounded in q.

Furthermore, the trace operator has a continuous right inverse.

3. In the following, we say that a function V satisfies condition

a) if V *is a real-valued* $\mathbf{C}^\infty$*-function in* $\overline{\mathbf{R}_+^n}$ *such that*

$$\partial^\alpha V(x) = O(|x|^{-\alpha})\,,\qquad as\quad |x|\rightarrow+\infty\,,\qquad \forall\alpha\,,\,|\alpha|\geq 0\,;$$

b) if $\forall\omega_0\in S_+^{n-1}$ *and* $\forall\varepsilon\in\mathbf{R}_+$, $\exists\delta(\omega_0,\varepsilon)$, $K(\omega_0,\varepsilon)\in\mathbf{R}_+$ *such that*

$$|\,\tilde V(\rho\,,\omega) - \tilde V(\sigma\,,\eta)\,| < \varepsilon\,,$$

$$\forall\,\rho\,,\sigma > K(\omega_0\,,\varepsilon)\quad,\quad \forall\omega\,,\eta\in S_+^{n-1}\cap S(\omega_0\,,\delta(\omega_0\,,\varepsilon))\,,$$

where $\tilde V$ *is the function* V *written in polar coordinates* (S_+^{n-1} denotes the $(n-1)$-dimensional unit half-sphere lying in $\overline{R_+^n}$).

Now we are able to state our main result.

THEOREM. *Let* $a_{ij}, b_i, c\ (i, j = 1, \cdots, n)$ *satisfy hypothesis* a) *and be such that*:

i) $a_{ij} = a_{ji}$;

ii) a_{ij} *satisfy hypothesis b)*;

iii) $\sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\,\xi_j > k\,|\,\xi\,|^2, \forall\xi \in \mathbf{R}^n, \forall x \in \overline{\mathbf{R}^n_+}$, *where* $k \in \mathbf{R}_+$.

Let $s \geq 2$. *Then, if* $f \in \mathrm{M}_q^{s-2}(\mathbf{R}^n_+), g \in \mathrm{M}_q^{s-\alpha-1/2}(\mathbf{R}^{n-1})$, *there exists* $q_0 \in \mathbf{R}_-$, *such that, if* $\operatorname{Re} q^2 \leq q_0$, *the problem*

$$\begin{cases} \left(\sum_{i,j=1}^{n} a_{ij}(x)\,\partial^2_{ij} + \sum_{i=1}^{n} b_i(x)\,\partial_i + c(x) + q^2 + x\cdot\nabla\right) u = f, & \text{in } \mathbf{R}^n_+, \\ \gamma_0\left(\alpha \sum_{j=1}^{n} a_{nj}(x)\,\partial_j + \beta\right) u = g, & \text{in } \mathbf{R}^{n-1}, \end{cases}$$

has a unique solution in $\mathrm{M}_q^s(\mathbf{R}^n)$. *Here* $\alpha, \beta \in \mathbf{R}, \alpha^2 + \beta^2 > 0, \alpha \in \{0, 1\}$.

Furhermore, a priori estimates hold true, with constants uniformly bounded in q.

To prove the theorem, we first suppose a_{ij}, b_i, c are constants. In this case, a priori estimates are obtained directly. Existence is obtained proving that the range of the operator is everywhere dense. This is accomplished by reducing our problem to the one studied in [4]. In this case exponential weights give no trouble.

Then, the general result is obtained by localizing and then gluing up the local solutions as in [1].

4. *Remarks.*

a) We use the complex parameter q for applications in parabolic problems (modulo a Laplace transformation) and also to obtain an existence result for a problem which is not Fredholm if arbitrary lower order terms are considered.

b) We note that the inequality in $\mathrm{L}^2(\mathbf{R}^n)$

$$|\,q\,|^2\,\|\,u\,\|_0 + \|\,x\cdot\nabla u\,\|_0 \leq \mathrm{C}\,\|\,\mathrm{A}_q\,u\,\|_0$$

is false; hence,

$$\mathscr{D}_{\mathrm{L}^2}(\mathrm{A}_q) \neq \{u \in \mathrm{L}^2(\mathbf{R}^n)\,;\, x\cdot\nabla u \in \mathrm{L}^2(\mathbf{R}^n)\}\,;$$

this fact compels us to consider A_q as a whole.

c) Among the functions satisfying condition *b*) in 3. there are:

- functions which have a uniform limit at infinity;
- functions which do not depend on the radial variable outside of a compact set.

Some hypotheses related to this one are formulated by Bagirov [2].

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