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A Remark on a Problem of Asplund

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Analisi funzionale. — *A Remark on a Problem of Asplund.* Nota [(*)] di SIMEON REICH, presentata dal Socio G. SANSONE.

RIASSUNTO. — Si dimostrano due risultati nella teoria di approssimazione.

The main purpose of this note is to present a simple proof of a result on the almost everywhere differentiability of the nearest point map in finite-dimensional spaces (Theorem 1). This result is due to Abatzoglou [1] who used a different (although similar) argument. It partially solves a problem of Asplund [4]. We also use part of our argument to provide a simple proof of a result on the convexity of suns (Theorem 2). This result is due to Klee [5] and to Efimov and Stechkin (see [7] and [3]).

Let E be a real Banach space. Recall that the subdifferential of a function $f: \mathrm{E} \to \mathrm{R}$ is defined by $\partial f(x) = \{x^* \in \mathrm{E}^* : f(y) - f(x) \geq (y - x, x^*)$ for all $y \in \mathrm{E}\}$. In the sequel we will use the following fact: if the subdifferential of f is nonempty for all x in E, then f must be convex. Indeed, if $w = tx + (1-t)y$ with $0 \leq t \leq 1$, then for each $x^* \in \partial f(w)$, $f(x) - f(w) \geq (1-t)(x-y, x^*)$, $f(y) - f(w) \geq t(y - x, x^*)$, and $tf(x) + (1-t)f(y) - f(w) \geq 0$.

Recall also that the duality map $\mathrm{J} : \mathrm{E} \to 2^{\mathrm{E}^*}$ is the subdifferential of $f(x) = \frac{1}{2}|x|^2$, and that if K is a subset of E, then the nearest point map P is defined by $\mathrm{P}x = \{y \in \mathrm{K} : |x - y| = d(x, \mathrm{K})\}$. If $\mathrm{P}x$ is nonempty for each x in E, then K is said to be proximinal. It is called a sun if $z \in \mathrm{P}x$ implies that $z \in \mathrm{P}(z + t(x - z))$ for all $t \geq 0$. J is single-valued if and only if E is smooth.

THEOREM 1. *Let* E *be finite-dimensional and suppose that the Fréchet derivative* J′ *of* J *exists and satisfies* $m|h|^2 \leq (\mathrm{J}'(x)h, h) \leq \mathrm{M}|h|^2$ *for some positive constants* m *and* M. *Let* K *be a closed subset of* E, *and let* $p: \mathrm{E} \to \mathrm{K}$ *be a selection of the nearest point map. Then* p *is Fréchet differentiable almost everywhere.*

Proof. Let $\mathrm{C} = \mathrm{M}/m$, and define $f: \mathrm{E} \to \mathrm{R}$ and $g: \mathrm{E} \to \mathrm{E}^*$ by $f(x) = \frac{1}{2}\mathrm{C}|x|^2 - \frac{1}{2}|x - p(x)|^2$ and $g(x) = \mathrm{CJ}x - \mathrm{J}(x - p(x))$. We have

$$(\mathrm{J}(x+h) - \mathrm{J}x, h) = \int_0^1 (\mathrm{J}'(x + th)h, h)\,\mathrm{d}t$$

(*) Pervenuta all'Accademia il 18 agosto 1979.

and

$$|x+y|^2 - |x|^2 - 2(y, Jx) = 2\int_0^1 (y, J(x+ty) - Jx)\,dt.$$

Therefore $m|x-y|^2 \leq (Jx - Jy, x-y) \leq M|x-y|^2$ and $m|y|^2 \leq \leq |x+y|^2 - |x|^2 - 2(y, Jx) \leq M|y|^2$. Consequently, $C|x+y|^2 - C|x|^2 - 2C(y, Jx) \geq Cm|y|^2 = M|y|^2 \geq |x-p(x)+y|^2 - |x-p(x)|^2 - 2(y, J(x-p(x))) \geq |x+y-p(x+y)|^2 - |x-p(x)|^2 - 2(y, J(x-p(x)))$. In other words, $f(x+y) - f(x) \geq (y, g(x))$. Thus $g(x)$ belongs to $\partial f(x)$ for all x in E, and f is convex.

By a result of Alexandrov [2], f is twice differentiable almost everywhere. If x is such a point of smoothness, then $f'(x) = g(x)$, and it can be shown that g is Fréchet differentiable at x. Since $p(x) = x + J^{-1}(g(x) - CJx)$, the result follows.

THEOREM 2. *A proximinal sun in a smooth Banach space is convex.*

Proof. Let $K \subset E$ be a proximinal sun, and let $p: E \to K$ be a selection of the nearest point map. Since K is a sun and J is single-valued,

$$(z - p(x), J(x - p(x))) \leq 0 \qquad \text{for all } z \text{ in } K.$$

Therefore $\frac{1}{2}|y-p(y)|^2 - \frac{1}{2}|x-p(x)|^2 \geq (y-x+p(x)-p(y), J(x-p(x))) = = (y-x, J(x-p(x))) + (p(x)-p(y), J(x-p(x))) \geq (y-x, J(x-p(x)))$. In other words, $J(x-p(x))$ belongs to the subdifferential of $f(x) = \frac{1}{2}d(x, K)^2$ for all x in E. Hence f is convex and $K = \{x \in E : f(x) = 0\}$ is convex too.

Remark. A similar argument shows that in Section 7 of [6] (where E may be a general Banach space), V must be convex. See the paper entitled "A general convergence principle in nonlinear functional analysis" by R. E. Bruck and the author.

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