Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Giovanni Prouse

On an inequality related to the motion, in any dimension, of viscous, incompressible fluids

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **67** (1979), n.3-4, p. 191–196.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1979_8_67_3-4_191_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1979.

Analisi matematica. — *On an inequality related to the motion, in any dimension, of viscous, incompressible fluids* (*). Nota I (**) di GIOVANNI PROUSE, presentata dal Socio L. AMERIO.

RIASSUNTO. — Si considera una disequazione associata al moto di un fluido viscoso incomprimibile in un numero qualsiasi di dimensioni e si enuncia un teorema di esistenza, unicità e dipendenza continua dai dati per una soluzione « debole » ed una soluzione « forte » di un classico problema di Cauchy-Dirichlet.

1. It is well known that, except in the case of plane motion, no global existence and uniqueness theorem has as yet been proved for the solution of the classical Cauchy-Dirichlet problem associated to the Navier–Stokes equations, corresponding to the motion of a viscous, incompressible fluid filling a vessel Ω, with given initial velocity $\vec{u}(x, 0)$.

This circumstance could either be due to the fact that the analytical tools employed for the study of this problem are inadequate, or else that the equations considered do not represent in a completely satisfactory way the motion of a viscous incompressible fluid.

In the present note and in the following one we want to investigate one aspect of the second alternative and consider, in particular, an inequality which is associated in a natural way to the Navier-Stokes equations; we shall then prove that the global Cauchy-Dirichlet problem is well posed for this inequality, in any number of dimensions, both for " weak " and for " strong " solutions.

Observe, to begin with, that the validity of the Navier-Stokes equations is obviously subject to the condition that the velocity of the fluid never reach a value close to that of the velocity c of light; in the latter case, in fact, the Navier-Stokes equations would have to be substituted by relativistic equations. If therefore we fix $c' < c$ appropriately, and give the velocity at all points of the fluid at the initial time $t = 0$ $(\vec{u}(x, 0) = \vec{\alpha}(x), x \in \Omega, |\vec{\alpha}| \leq c')$, we may, in a first appoximation admit that the solutions of the Navier-Stokes equations represent the motion of the fluid in Ω during the time interval $0 \leq t < \bar{t}$, with

$$(1.1) \qquad \bar{t} = \operatorname{Sup}\{t \geq 0 ; |\vec{u}(x, t)| \leq c' \quad \text{a.e. in} \quad \Omega \times [0, t]\}.$$

Considering any subsequent time interval $\bar{t} \leq t \leq t'$, the velocity, is, by (1.1), such that $|\vec{u}(x, t)| > c'$ on a set of positive measure $\subset \Omega \times (\bar{t}, t)$: hence, $\vec{u}$ loses, for $t > \bar{t}$ its physical meaning.

(*) Lavoro eseguito nell'ambito dell'attività del Gruppo Nazionale per l'Analisi Funzionale e le sue applicazioni del C.N.R.

(**) Pervenuta all'Accademia il 29 agosto 1979.

By what has been said above, we can, in the description of the motion of a viscous incompressible fluid, substitute to the Navier-Stokes equations any other relation which coincides with such equations when $0 \leq t \leq \bar{t}$; for $t > \bar{t}$ this relation may also differ from the Navier-Stokes equations because these do not, anyway, have a physical interpretation.

2. Let us now give a precise formulation of the problem outlined in the preceding paragraph.

Denoting by Ω a bounded, open set $\subset \mathbf{R}^m$ and by Γ its boundary, the Navier-Stokes equations which govern the motion in Ω of an incompressible fluid of unit density and viscosity μ are

$$\left\{ \begin{aligned} & \frac{\partial \vec{u}}{\partial t} - \mu \Delta \vec{u} + (\vec{u} \cdot \operatorname{grad}) \vec{u} + \operatorname{grad} p = \vec{f} \\ & \operatorname{div} \vec{u} = 0 \end{aligned} \right. \tag{2.1}$$

where $\vec{u} = \{u_1, u_2, \cdots, u_m\}$ is the velocity, p the pressure, $\vec{f}$ the external force.

To equations (2.1) we shall associate the initial and boundary conditions

$$\vec{u}(x, 0) = \vec{\alpha}(x) \qquad (x \in \Omega) \tag{2.2}$$

$$\vec{u}(x, t) = 0 \qquad (0 \leq t \leq T ; x \in \Gamma) \tag{2.3}$$

We introduce the following notations:

$\mathcal{N} = \{\vec{v}(x) : v_j \in \mathcal{D}(\Omega)\ (j = 1, 0 \cdots, m), \operatorname{div} \vec{v} = 0\}$;

N^0 = closure of $\mathcal{N}$ in $L^2(\Omega)$; $(\vec{u}, \vec{v})_{N^0} = (\vec{u}, \vec{v})_{L^2(\Omega)} = \int_\Omega \sum_{j=1}^m u_j v_j \, d\Omega$;

N^1 = closure of $\mathcal{N}$ in $H^1(\Omega)$; $(\vec{u}, \vec{v})_{N^1} = (\vec{u}, \vec{v})_{H_0^1(\Omega)} = \int_\Omega \sum_{k,j=1}^m \frac{\partial u_j}{\partial x_k} \cdot \frac{\partial v_j}{\partial x_k} \, d\Omega$;

N^s = closure of $\mathcal{N}$ in $H^s(\Omega)$ $(s \geq 0)$; $(\vec{u}, \vec{v})_{N^s} = (\vec{u}, \vec{v})_{H_0^s(\Omega)}$;

$K = \{\vec{v} \in N^0 : |\vec{v}| \leq c$ a.e. in $\Omega\}$; the set K is obviously convex and closed in N^0;

$(N^s)'$ = dual space of N^s ; $(N^0)' = N^0$;

$b(\vec{u}, \vec{v}, \vec{w}) = ((\vec{u} \cdot \operatorname{grad}) \vec{v}, \vec{w})_{N^0}$;

$\langle u, v \rangle$ = duality between $(N^1)'$ and N^1.

We shall say that $\vec{u}(t) = \{\vec{u}(x,t); x \in \Omega\}$ is a *weak solution* in $[0,T]$ of the Navier-Stokes equations satisfying (2.2), (2.3) if:

a_1') $\vec{u}(t) \in L^\infty(0,T;N^0) \cap L^2(0,T;N^1)$;

b_1') $\vec{u}(t)$ satisfies the equation

$$\int_0^T \{-(\vec{u},\vec{\varphi}')_{N^0} + \mu(\vec{u},\vec{\varphi})_{N^1} + b(\vec{u},\vec{u},\vec{\varphi}) - \langle \vec{f},\vec{\varphi}\rangle\}\, dt = (\vec{\alpha},\vec{\varphi}(0))_{N^0}$$

$\forall \vec{\varphi}(t) \in L^2(0,T;N^s)$, with $\vec{\varphi}'(t) \in L^2(0,T;N^0)$, $\vec{\varphi}(T) = 0$, $s = m/2$.

We shall say that $\vec{u}(t)$ is a *strong solution* in $[0,T]$ of the Navier-Stokes equations satisfying (2.2), (2.3) if:

a_1'') $\vec{u}(t) \in L^\infty(0,T;N^1)$, $\vec{u}'(t) \in L^2(0,T;N^0)$, $\Delta\vec{u}(t) \in L^2(0,T;N^0)$, $\vec{u}(0) = \vec{\alpha}$

b_1'') $\vec{u}(t)$ satisfies the equation

$$\int_0^T \{(\vec{u}' - \mu\Delta\vec{u} - \vec{f},\vec{\varphi})_{N^0} + b(\vec{u},\vec{u},\vec{\varphi})\}\, dt = 0 \qquad \forall \vec{\varphi}(t) \in L^2(0,T;N^s).$$

We shall say that $\vec{v}(t)$ is a *weak solution* in $[0,T]$ of the Navier-Stokes inequalities relative to the convex set K, satisying (2.2), (2.3) if:

a_2') $\vec{v}(t) \in L^\infty(0,T;N^0) \cap L^2(0,T;K \cap N^1)$;

b_2') $\vec{v}(t)$ satisfies a.e. in $[0,T]$ the inequality

$$\frac{1}{2}\|\vec{v}(t) - \vec{\varphi}(t)\|^2_{N^0} + \int_0^t \{(\vec{\varphi}',\vec{v}-\vec{\varphi})_{N^0} + \mu(\vec{v},\vec{v}-\vec{\varphi})_{N^1} - \langle\vec{f},\vec{v}-\vec{\varphi}\rangle +$$
$$+ b(\vec{v},\vec{v},\vec{v}-\vec{\varphi})\}\, d\eta - \frac{1}{2}\|\vec{\alpha} - \vec{\varphi}(0)\|^2_{N^0} \leq 0$$

$\forall \vec{\varphi}(t) \in L^2(0,T;K \cap N^1)$, with $\vec{\varphi}'(t) \in L^2(0,T;N^0)$.

We shall say that $\vec{v}(t)$ is a *strong solution* in $[0,T]$ of the Navier-Stokes inequalities relative to the convex set K, satisfying (2.2), (2.3) if:

a_2'') $\vec{v}(t) \in L^\infty(0,T;K \cap N^1)$, $\Delta\vec{v}(t) \in L^2(0,T;N^0)$;

b_2'') $\vec{v}(t)$ satisfies the inequality, a.e. in $[0,T]$,

$$\frac{1}{2}\|\vec{v}(t) - \vec{\varphi}(t)\|^2_{N^0} + \int_0^t \{(\vec{\varphi}',\vec{v}-\vec{\varphi})_{N^0} - \mu(\Delta\vec{v},\vec{v}-\vec{\varphi})_{N^0} - \langle\vec{f},\vec{v}-\vec{\varphi}\rangle +$$
$$+ b(\vec{v},\vec{v},\vec{v}-\vec{\varphi})\}\, d\eta - \frac{1}{2}\|\vec{\alpha} - \vec{\varphi}(0)\|^2_{N^0} \leq 0$$

$\forall \vec{\varphi}(t) \in L^2(0,T;K)$ with $\vec{\varphi}'(t) \in L^2(0,T;N^0)$.

Let us now investigate the *relationships between the solutions of the equations and of the inequalities.*

Assume, at first, that $\vec{v}(t)$ satisfies $a_2')$ and $b_2')$ and, moreover, that there exists $\bar{t} > 0$ such that

$$|\vec{v}(x,t)| \leq c' < c \qquad \text{a.e. in} \quad \Omega[0,\bar{t}]. \tag{2.4}$$

We shall show that $\vec{v}(t)$ is then a weak solution of the Navier-Stokes equations in $[0,\bar{t}]$.

Let $\{\vec{v}_j(t)\}$ be a regularising sequence associated to $\vec{v}$, such that

$$\vec{v}_j(t) \in H^1(0,T;N^1) \quad , \quad \vec{v}_j(t) \xrightarrow[L^2(0,T;N^1)]{} \vec{v}(t) \quad , \quad \vec{v}_j(0) = \vec{\alpha}, \tag{2.5}$$

$$\int_0^t (\vec{v}_j', \vec{v} - \vec{v}_j)_{N^0}\, d\eta \geq 0 \qquad \forall t \in [0,T], \tag{2.6}$$

$$|\vec{v}_j(x,t)| \leq c' \quad \text{a.e. in } \Omega\times[0,\bar{t}], \ |\vec{v}_j(x,t)| \leq c \quad \text{a.e. in } \Omega\times[0,T]. \tag{2.7}$$

It can be proved (see, for instance, Lions [1], ch. 2, th. 9.1) that, under the assumptions made, such a sequence exists (1). Let moreover, $\vec{\psi}(t)$ be an arbitrary function $\in L^2(0,T,\mathcal{N})$, with $\vec{\psi}'(t) \in L^2(0,T;N^0)$, $\vec{\psi}(t) = 0$ when $t \geq \bar{t}$ and set in $b_2')$

$$\vec{\varphi} = \vec{v}_j - \lambda\vec{\psi} \quad , \quad \lambda > 0 \tag{2.8}$$

choosing λ so small that $|\vec{\varphi}(x,t)| \leq c$ in $\Omega\times[0,\bar{t}]$; this is obviously possible, since $|\vec{v}_j| \leq c' < c$ in $\Omega\times[0,\bar{t}]$.

We have then, , a.c. in $[0,T]$,

$$\begin{gathered}\tfrac{1}{2}\|\vec{v}(t) - \vec{v}_j(t) + \lambda\vec{\psi}(t)\|^2_{N^0} + \int_0^t \{(\vec{v}_j' - \lambda\vec{\psi}', \vec{v} - \vec{v}_j + \lambda\vec{\psi})_{N^0} + \\ + \mu(\vec{v}, \vec{v} - \vec{v}_j + \lambda\vec{\psi})_{N^1} - \langle\vec{f}, \vec{v} - \vec{v}_j + \lambda\vec{\psi}\rangle + b(\vec{v}, \vec{v}, \vec{v} - \vec{v}_j + \lambda\vec{\psi})\}\, d\eta - \\ - \tfrac{1}{2}\|\lambda\vec{\psi}(0)\|^2_{N^0} \leq 0\end{gathered} \tag{2.9}$$

and consequently, by (2.6),

$$\begin{gathered}\int_0^t \{(\vec{v}_j', \lambda\vec{\psi})_{N^0} - (\lambda\vec{\psi}', \vec{v} - \vec{v}_j + \lambda\vec{\psi})_{N^0} + \mu(\vec{v}, \vec{v} - \vec{v}_j + \lambda\vec{\psi})_{N^1} - \\ - \langle\vec{f}, \vec{v} - \vec{v}_j + \lambda\vec{\psi}\rangle + b(\vec{v}, \vec{v}, \vec{v} - \vec{v}_j + \lambda\vec{\psi})\}\, d\eta - \tfrac{1}{2}\|\lambda\vec{\psi}(0)\|^2_{N^0} \leq 0 \quad \forall t \in [0,T].\end{gathered} \tag{2.10}$$

(1) The functions $\vec{v}_j$ can be explicitely defined by

$$\frac{1}{j}\vec{v}' + \vec{v}_j = \vec{v} \quad , \quad \vec{v}_j(0) = \vec{\alpha}.$$

Hence we can set in (2.10) $t = \bar{t}$ and integrate by parts, thus obtaining, since $\vec{v}_j(0) = \vec{\alpha}$, $\vec{\psi}(\bar{t}) = 0$,

$$(2.11) \quad \int_0^{\bar{t}} \{-(\vec{v}_j, \lambda\vec{\psi}')_{N^0} - (\lambda\vec{\psi}', \vec{v} - \vec{v}_j + \lambda\vec{\psi})_{N^0} + \mu(\vec{v}, \vec{v} - \vec{v}_j + \lambda\vec{\psi})_{N^1} -$$

$$- \langle \vec{f}, \vec{v} - \vec{v}_j + \lambda\vec{\psi}\rangle + b(\vec{v}, \vec{v}, \vec{v} - \vec{v}_j + \lambda\vec{\psi})\}\, dt - (\vec{v}_j(0), \lambda\vec{\psi}(0))_{N^0} -$$

$$- \tfrac{1}{2} \| \lambda\vec{\psi}(0) \|^2_{N^0} \leq 0.$$

It follows, letting $j \to \infty$ and then dividing by λ,

$$(2.12) \quad \int_0^{\bar{t}} \{-(\vec{v}, \vec{\psi}')_{N^0} - \lambda(\vec{\psi}', \vec{\psi})_{N^0} + \mu(\vec{v}, \vec{\psi})_{N^1} - \langle \vec{f}, \vec{\psi}\rangle +$$

$$+ b(\vec{v}, \vec{v}, \vec{\psi})\}\, dt - (\vec{\alpha}, \vec{\psi}(0))_{N^0} - \frac{\lambda}{2} \| \vec{\psi}(0) \|^2_{N_0} \leq 0 .$$

We now let $\lambda \to 0$ and obtain

$$(2.13) \quad \int_0^{\bar{t}} \{-(\vec{v}, \vec{\psi}')_{N^0} + \mu(\vec{v}, \vec{\psi})_{N^1} - \langle \vec{f}, \vec{\psi}\rangle + b(\vec{v}, \vec{v}, \vec{\psi})\}\, dt \leq (\vec{\alpha}, \vec{\psi}(0))_{N^0}.$$

Hence, being $\vec{\psi}$ arbitrary,

$$(2.14) \quad \int_0^{\bar{t}} \{-(\vec{v}, \vec{\psi}')_{N^0} + \mu(\vec{v}, \vec{\psi})_{N^1} - \langle \vec{f}, \vec{\psi}\rangle + b(\vec{v}, \vec{v}, \vec{\psi})\}\, dt = (\vec{\alpha}, \vec{\psi}(0))_{N^0}.$$

Since $L^2(0, \bar{t}; \mathcal{N})$ is dense in $L^2(0, \bar{t}; N^s)$, (2.14) hold also $\forall \vec{\psi}(t) \in L^2(0, \bar{t}; N^s)$ such that $\vec{\psi}'(t) \in L^2(0, \bar{t}; N^0)$, $\vec{\psi}(\bar{t}) = 0$; $\vec{v}(t)$ satisfies therefore also condition b_1') and is a weak solution in $[0, \bar{t}]$ of the Navier-Stokes equations.

Assume now that $\vec{v}(t)$ satisfies a_2''), b_2'') and that also (2.4) holds; assume, moreover, that $\vec{f}(t) \in L^2(0, T; N^0)$, that $m \leq 4$ and that Ω is "sufficiently smooth". Then $\vec{v}(t)$ is a solution of the Navier-Stokes equations a.e. in $\Omega \times [0, \bar{t}]$.

By what been proved above, we have, in fact,

$$(2.15) \quad \int_0^{\bar{t}} \{-(\vec{v}, \vec{\psi}')_{N^0} - \mu(\Delta\vec{v}, \vec{\psi})_{N^0} - (\vec{f}, \vec{\psi})_{N^0} + b(\vec{v}, \vec{v}, \vec{\psi})\}\, dt = 0$$

$\forall \vec{\psi}(t) \in L^2(0, t; N^0)$, with $\vec{\psi}'(t) \in L^2(0, \bar{t}; N^0)$, $\vec{\psi}(\bar{t}) = \vec{\psi}(0) = 0$. Since, on the other hand,

$$(2.16) \qquad \left| \int_0^{\bar{t}} \{\mu\,(\Delta\vec{v}\,,\vec{\psi})_{N^0} + (\vec{f}\,,\vec{\psi})_{N^0} - b\,(\vec{v}\,,\vec{v}\,,\vec{\psi})\}\,dt \right| \leq M\, \|\vec{\psi}\|_{L^2(0,T;N^0)}$$

it follows directly from (2.14) that condition b_1'') holds.

Bearing in mind the observations made in § 1, we may substitute, for the description of the motion in Ω of a viscous incompressible fluid, to the Navier-Stokes equations the corresponding inequalities. If, in fact, we choose c' sufficiently close to c, then neither the solutions of the Navier-Stokes equations or inequalities have a physical meaning for $t > \bar{t}$, while, when $0 \leq t \leq \bar{t}$, the solutions of the equations and inequalities coincide, as has been proved above.

In what follows we shall therefore consider the Navier-Stokes inequalities and prove that problem (2.2), (2.3) is well posed both in the weak and in the strong sense.

OBSERVATION 1. *If* $\vec{v}\,(t)$ *satisfies* a_2'), b_2'), *then* $\vec{v}\,(t) \in C^0([0\,,T]\,;N^0)$. Let, in fact, $\vec{v}_j\,(t)$ be the regularising sequence defined at page 194 and set in b_2') $\vec{\psi}\,(t) = \vec{v}_j\,(t)$; we have then, bearing in mind (2.6),

$$\frac{1}{2}\,\|\vec{v}\,(t) - \vec{v}_j\,(t)\|^2_{N^0} \leq \int_0^t (-\,\mu\,(\vec{v}\,,\vec{v} - \vec{v}_j)_{N^1} + \langle\vec{f}\,,\vec{v} - \vec{v}_j\rangle + b\,(\vec{v}\,,\vec{v}\,,\vec{v} - \vec{v}_j)\}\,d\eta .$$

Since the right hand side of (2.17) tends to zero uniformly with respect to t when $j \to \infty$, we can conclude that $\vec{v}\,(t) \in C^0([0\,,T]\,;N^0)$.

OBSERVATION 2. The case $m = 2$ has been studied by Biroli [2] who has proved that problem a_2'), b_2') is well posed $\forall$ set K which is closed and convex in N_3, with $0 \subset K$.

BIBLIOGRAPHY

[1] J. L. LIONS (1969) – *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, Paris.

[2] M. BIROLI (1972) – *Sur l'inequation d'évolution de Navier-Stokes*, note I, II, III. « Rend. Acc. Naz. Lincei », VII, vol. LII.