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# Rendiconti

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# Asymptotic behaviour of solutions of a non linear model of population dynamics

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**Analisi matematica.** — *Asymptotic behaviour of solutions of a non linear model of population dynamics.* Nota (*) di EUGENIO SINESTRARI (**), presentata dal Socio G. SANSONE.

RIASSUNTO. — Si studia l'esistenza e l'unicità delle soluzioni stazionarie di una equazione funzionale non lineare proveniente dalla dinamica di popolazione e se ne dimostra la stabilità.

## 1. INTRODUCTION

In a previous paper [1] we introduced a non linear version of the classical Lotka-Von Foerster equations which leads to the problem of finding $u(a,t) \geq 0$ such that

$$(\mathrm{P}) \quad \begin{cases} \mathrm{D}u(a,t) = -m(a,u(a,t))\,u(a,t) & 0 \leq a < \omega,\ t \geq 0. \\ u(0,t) = \displaystyle\int_0^\omega b(a,u(a,t))\,u(a,t)\,\mathrm{d}a & t > 0 \\ u(a,0) = p(a) & 0 \leq a < \omega \end{cases}$$

where $\mathrm{D}u(a,t) = \lim\limits_{h\to 0} \dfrac{u(a+h,t+h)-u(a,t)}{h}$, $m$, $b$ and $p$ are given functions and $\omega$ is a given positive number. We will set $\mathrm{A} = [0,\omega[$ and $u(0,t) = \mathrm{B}(t)$. For the physical meaning of the problem we refer to [1].

Global existence and uniqueness results for the solution to (P) were obtained under the following assumptions:

($m_1$) $m(a,u)$ is continuous and non negative on $\mathrm{A}\times\mathrm{R}_+$

($m_2$) $u \to m(a,u)$ is non decreasing

($m_3$) given $a\in\mathrm{A}$ and $\delta>0$, $\lim\limits_{k\to\omega}\displaystyle\int_a^k m(x,u)\,\mathrm{d}x = +\infty$ uniformly for $u\leq\delta$

($b_1$) $b(a,u)$ is continuous and non negative on $\mathrm{A}\times\mathrm{R}_+$ and $b^* = \sup b < +\infty$

($b_2$) given $\delta>0$, there exists $\mathrm{L} = \mathrm{L}(\delta)$ such that $|b(a,u') - b(a,u'')| \leq \mathrm{L}\,|u'-u''|$ for $a\in\mathrm{A}$ and $0\leq u', u''\leq\delta$.

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$(p_1)$ $p$ is piecewise continuous and non negative on A

$(p_2)$ $p^* = \sup p$ and $P_0 = \int_0^\omega p(a)\, da$ are finite.

In this paper we need also the following conditions:

$(m_4)$ $m_u(a, u)$ exists continuous on $A \times R_+$

$(b_3)$ $b_u(a, u) \leq 0$ and is continuous on $A \times R_+$

## 2. STATIONARY SOLUTIONS

In this section we want to give conditions for the existence of stationary solutions of problem (P): they correspond to the equilibrium age distributions for the population.

DEFINITION 1. A function $a \to p(a)$ from A to $R_+$ is a stationary solution of (P) if the following equations are satisfied:

$$(S) \quad \begin{cases} p'(a) = -m(a, p(a))\, p(a) & a \in A \\ p(0) = \int_0^\omega b(a, p(a))\, p(a)\, da \end{cases}$$

We shall assume $p(0) = \varepsilon > 0$ so that $p \not\equiv 0$; in [1] we proved the existence and uniqueness of the solution of $(S_1)$ in A, when $p(0)$ is a prescribed positive number.

PROPOSITION 1. *Let* $p_\varepsilon : A \to R_+$ *be a solution to*

$$(1) \quad \begin{cases} p'(a) = -m(a, p(a))\, p(a) & a \in A \\ p(0) = \varepsilon \end{cases}$$

*then* $p_\varepsilon$ *is a stationary solution to* (P) *if and only if*

$$F(\varepsilon) = \int_0^\omega b(a, p_\varepsilon(a))\, p_\varepsilon(a) \cdot \varepsilon^{-1}\, da = 1.$$

LEMMA 1. *The function* $\varepsilon \to F(\varepsilon)$ *is non increasing on* $]0, +\infty[$ *so that there exist* $F(0^+)$ *and* $F(\infty)$. *If* $u \to m(a, u)$ *and* $u \to b(a, u)$ *are strictly monotonic then* F *is strictly decreasing.*

*Proof.* If $0 < \varepsilon < \varepsilon'$, then $p_\varepsilon < p_{\varepsilon'}$. From (1) we deduce that $p_\varepsilon(a) = \varepsilon \exp\left(-\int_0^a m(x, p_\varepsilon(x))\, dx\right)$ hence $\varepsilon \to p_\varepsilon(a)\varepsilon^{-1}$ is non increasing by $(m_2)$;

from $(b_3)$ it follows that $\varepsilon \to F(\varepsilon)$ is non increasing. The last part of the theorem is obvious.

We give now conditions for the existence and uniqueness of stationary solutions to (P).

THEOREM 1. *If there is a stationary solution then* $F(0^+) \geq 1$ *and* $F(\infty) \leq 1$. *Conversely let* $F(0^+) > 1$ *and* $F(\infty) < 1$: *then there exists stationary solutions; if in addition* $u \to m(a, u)$ *and* $u \to b(a, u)$ *are strictly monotonic then the stationary solution is unique.*

The proof is a consequence of lemma 1.

In many practical situations, $p_\varepsilon(a)$ can be written in closed form so that $F(0^+)$ and $F(\infty)$ can be computed explicitly.

## 3. STABILITY OF STATIONARY SOLUTIONS.

In this section we shall suppose that there exists a stationary solution $p_\varepsilon(a)$ to (P) and we want to investigate the asymptotic behaviour of a solution to (P), close to $p_\varepsilon$ at $t = 0$.

Let $u : A \times R_+ \to R_+$ be a solution to (P). Set

$$\begin{cases} r(a, t) = u(a, t) - p_\varepsilon(a) \\ R(t) \quad = r(0, t) = B(t) - \varepsilon \\ \rho(a) \quad = r(a, 0) = p(a) - p_\varepsilon(a). \end{cases}$$

With these notations and taking into account (P) and (S) we can write the equations satisfied by $r(a, t)$:

$$(R) \begin{cases} Dr(a, t) = -\lambda(a, t)\, r(a, t) & a \in A, t \geq 0 \\ r(0, t) = \int_0^\omega H(a, t)\, r(a, t)\, da & t > 0 \\ r(a, 0) = \rho(a) & a \in A \end{cases}$$

where

$$\begin{cases} \lambda(a, t) = m_u(a, p_\varepsilon(a) + \theta r(a, t))\, p_\varepsilon(a) + m(a, p_\varepsilon(a) + r(a, t)) \\ H(a, t) = b_u(a, p_\varepsilon(a) + \theta' r(a, t))\, p_\varepsilon(a) + b(a, p_\varepsilon(a) + r(a, t)) \end{cases}$$

with suitable $\theta, \theta' \in [0, 1]$.

From (R) is follows that

$$r(a,t)=\begin{cases} R(t-a)\exp\left(-\int_0^a \lambda(x,t-a+x)\,dx\right) & a<t \\ \rho(a-t)\exp\left(-\int_0^t \lambda(a-t+x,x)\,dx\right) & a\geq t \end{cases}$$

hence from ($R_2$) we obtain an integral equation which is satisfied by $R(t)$:

$$R(t)=\int_0^t K(a,t)\,R(t-a)\,da+f(t) \qquad t\geq 0 \tag{2}$$

where:

$$K(a,t)=\begin{cases} H(a,t)\exp\left(-\int_0^a \lambda(x,t-a+x)\,dx\right) & a\leq t\,,\ a<\omega \\ 0 & a\geq t\,,\ a\geq\omega \end{cases}$$

and

$$f(t)=\begin{cases} \int_t^\omega H(a,t)\exp\left(-\int_0^t \lambda(a-t+x,x)\,dx\right)\rho(a-t)\,da\,, & 0\leq t\leq\omega \\ 0 & t\geq\omega\,. \end{cases}$$

We give now a theorem about the stability of a stationary solution to (P).

THEOREM 2. *Let $p_\varepsilon(a)$ be a stationary solution such that the following condition holds*:

$$\text{(L)}\qquad \int_0^\omega |\,b_u(a,p_\varepsilon(a))\,p_\varepsilon(a)+b(a,p_\varepsilon(a))\,|\exp\left(-\int_0^a (m_u(x,p_\varepsilon(x))\,p_\varepsilon(x)+ m(x,p_\varepsilon(x)))\,dx\right)da<1$$

*Then $p_\varepsilon$ is locally stable i.e. for each $\eta>0$ there exists $\delta_\eta$ such that if $u$ is a solution to* (P) *with $|u(a,0)-p_\varepsilon(a)|<\delta_\eta$ for $a\in A$ then $|u(a,t)-p_\varepsilon(a)|\leq\eta$ for $a\in A$ and each $t\geq 0$.*

Condition (L) is verified for example when $b_u(a,u)<0$ and $b_u(a,u)\,u+b(a,u)\geq 0$ or when $b$ does not depend on $u$ and for each $\bar{u}$, $m_u(a,\bar{u})\not\equiv 0$ on A. When $b_u\equiv m_u\equiv 0$, i.e. in the linear case, the behaviour of solutions is well known (see [2]).

*Proof.* Let us denote by L the integral in condition (L). From ($b_3$) and ($m_4$) it is not difficult to prove that given $\gamma \in ]\mathrm{L}, 1[$ there exists $\delta > 0$ such that if $t > \omega$ and $|\mathrm{R}(s)| < \delta$ for $s \in [0, t[$ then

$$\int_0^\omega |\mathrm{K}(a, t)| \, \mathrm{d}a < \gamma < 1 . \tag{3}$$

By using Theorem 5 of [1] it is possible to find a constant $c$ such that if $|\rho(a)| < 1$ then for $0 \leq t < \omega$

$$|\mathrm{R}(t)| \leq c \int_0^\omega |\rho(a)| \, \mathrm{d}a . \tag{4}$$

Let us choose $\eta < 1$. Then by taking $\delta_\eta = \min(\eta, \delta(\omega c)^{-1}, \eta(\omega c)^{-1})$ it can be proved by contradiction that (3) holds for every $t$. From this the conclusion follows easily.

We give now a last result which gives more precise informations about the behaviour of the birth rate $\mathrm{B}(t) = u(0, t)$ for large $t$.

THEOREM 3. *Let $p_\varepsilon$ be a stationary solution such that* (L) *holds; then there exist $c$ and $\delta > 0$ such that if $u$ is a solution to* (P) *with $|u(a, 0) - p_\varepsilon(a)| < \delta$ for $a \in \mathrm{A}$ then we have $|u(0, t) - \varepsilon| \sim \exp(-ct)$.*

*Proof.* Take $\gamma \in ]\mathrm{L}, 1[$ and choose $c > 0$ such that $\gamma \exp(c\omega) < 1$. Set $\mathrm{G}(t) = \mathrm{R}(t) \exp(ct)$. We must show that $\mathrm{G}(t)$ is bounded when $|\rho(a)| < \delta$ where $\delta$ is defined in the proof of Theorem 2. From (2) we get for $t \in [\omega, \mathrm{T}]$:

$$|\mathrm{G}(t)| \leq \sup_{t \in [0,\mathrm{T}]} |\mathrm{G}(t)| \int_0^\omega |\mathrm{K}(a, t)| \exp(ca) \, \mathrm{d}a \leq (\sup_{t \in [0,\omega]} |\mathrm{G}(t)| + \sup_{t \in [\omega,\mathrm{T}]} |\mathrm{G}(t)|)$$
$$\gamma \exp(c\omega) .$$

Hence

$$(1 - \gamma \exp(c\omega)) \sup_{t \in [\omega,\mathrm{T}]} |\mathrm{G}(t)| \leq \sup_{t \in [0,\omega]} |\mathrm{G}(t)| .$$

Letting $\mathrm{T} \to +\infty$, the conclusion follows.

## REFERENCES

[1] E. SINESTRARI – *A non linear functional renewal equation*, « Rend. Acc. Naz. Lincei ».
[2] F. HOPPENSTEADT (1975) – *Mathematical theories of Populations: Demographics, Genetics and Epidemics*, S.I.A.M., Philadelphia.