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Theorems A and B for henselian schemes

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Geometria algebrica. — *Theorems* A *and* B *for henselian schemes*. Nota [(*)] di SILVIO GRECO e ROSARIO STRANO, presentata dal Corrisp. E. MARCHIONNA.

RIASSUNTO. — Schema di una dimostrazione dei Teoremi A e B per fasci quasi coerenti su uno schema henseliano affine.

INTRODUCTION

Henselian schemes were introduced by Hironaka [15], and later on studied by Kurke, Pfister and Roczen [17], and independently in [8], [9] and by Mora [18]. They provide a good notion of "tubolar algebraic neighborhood" of a closed subvariety of an algebraic variety, as shown by Cox [3], [4].

In this note we outline a proof of the so called Theorems A and B for affine Hensel schemes, which show that quasi coherent sheaves over a Hensel scheme behave as they are expected to do. Applications of these theorems to equivalence of embeddings are given by Roczen [20]. The complete proofs, which are long and involved, will appear elsewhere.

1. PRELIMINARIES AND STATEMENTS OF THE MAIN RESULTS

A couple (A, a), consisting of a commutative ring A and of an ideal a of A is an H-*couple* (*Hensel couple*) if the Hensel Lemma holds for it (see [10], [20], [17]). To any couple (A, a) one can associate, in a canonical way, a Hensel couple ${}^h(A, a)$, called *henselization* (l. cit.). If B is an A-algebra we usually write ${}^hB$ in place of ${}^h(B, aB)$.

To any H-couple (A, a) one can associate the ringed space $(X, \mathbf{O}_X) = \mathrm{Sph}(A, a)$ (the *henselian spectrum* of (A, a)), and to any A-module M one can associate a quasi coherent sheaf $\tilde{M}$ over X, defined by: $\Gamma(X_f, \tilde{M}) = M \otimes_A {}^hA_f$, where $X_f = \mathrm{sph}({}^hA_f) \subset X$. Recall that $X = \mathrm{spec}(A/a)$ (as a topological space), and that $\mathbf{O}_X = \tilde{A}$.

A ringed space isomorphic to a henselian spectrum is an *affine Hensel scheme* (see [8], [17]).

THEOREM 1.1. (Theorem A). *Let* $X = \mathrm{sph}(A, a)$ *be an affine Hensel scheme, and let* $\mathbf{F}$ *be a quasi coherent sheaf over* X, *and put* $M = \Gamma(X, \mathbf{F})$. *Then* $\mathbf{F} = \tilde{M}$ (that is $\mathbf{F}$ is generated by its global sections).

THEOREM 1.2. (Theorem B). *Let* $X, \mathbf{F}$ *be as above. Then* $H^p(X, \mathbf{F}) = 0$ *for all* $p > 0$.

(*) Pervenuta all'Accademia il 13 agosto 1979.

Remarks 1.3. (i) The above results are known in many similar cases: see e.g. [7] or [14] for ordinary schemes, and [13], [12] for coherent sheaves over a formal scheme, and over a Stein space respectively.

(ii) One can deduce from 1.1 and 1.2 several results, by easy adaptments of the "ordinary" case. However we are not able to prove the fundamental Theorem on affine morphisms ([6], 9.1.10), but in this particular case:

COROLLARY 1.4. *Let* $f: X \to Y$ *be an affine integral morphism of Henselian schemes. Then if* Y *is affine also* X *is affine.*

2. AUXILIARY RESULTS

Let (A, a) be a Hensel couple, $X = \mathrm{sph}(A, a)$, $f, g \in A$ with $(f, g) A = A$. Put $R = {}^hA_f \otimes_A {}^hA_g$, $R_{Zar} = R_{1+aR}$, and let $\varphi: R_{zar} \to {}^hA_{fg}$, $\psi: R \to {}^hA_{fg}$ be the canonical morphisms.

THEOREM 2.1. φ *is an isomorphism.*

THEOREM 2.2. ψ *is surjective.*

To prove the above we need the following notion:

Definition 2.3. A ring A is AIC (*absolutely integrally closed*) if every monic polynomial in $A[X]$ has a root in A.

By general properties of H-couples (see [10]) one can show:

LEMMA 2.4. *If* A *is* AIC *and* Y *is a connected component of* X_f *then there is a multiplicative subset* $S \subset A$ *such that* (A_S, aA_S) *is an* H-*couple and* $Y = \mathrm{sph}(A_S)$.

THEOREM 2.5. *Assume* A *is* AIC *and let* $Y = \mathrm{sph}(A_S)$, $Z = \mathrm{sph}(A_T)$ *be connected components of* X_f, X_g *respectively* (see 2.4). *Then* ${}^hA_{ST} = A_{ST}$.

Proof of 2.1. (outline). By adapting an argument of M. Artin [1] one can reduce the problem to prove 2.5 when A is a domain. If $C = A_{ST}$, then it is sufficient to prove: (1) C/aC is connected, and (2) $aC \subset \mathrm{rad}(C)$. For (1) we need:

LEMMA 2.6. *There are an affine Hensel scheme* $W = \mathrm{sph}(B, aB)$ *and a monomorphism* $j: W \to X$ *such that* $j(W) = Y \cup Z$, *and* B *is an* AIC *domain.*

By Gruson [11] (or [17], 4.5.1) the scheme $\mathrm{spec}(B/aB)$ is simply connected, and from this one can deduce that it is connected. But $\mathrm{spec}(B/aB) = Y \cup Z = \mathrm{spec}(C/aC)$ (as topological spaces) and (1) follows. By a direct computation one can prove (2), and hence the Theorem.

Theorems 2.2 and 2.5 can be deduced from 2.1, by using similar tecniques.

3. PROOF OF THEOREM A

Notations as in 2.1. It is sufficient to show that $\mathrm{I} = \{f \in \mathrm{A} ; \mathbf{F}_{|\mathrm{X}_f}$ is generated by $\Gamma(\mathrm{X}_f, \mathbf{F})\}$ is an ideal of A.

The hard part is to show that if $f_1, f_2 \in \mathrm{I}$, then $f_1 + f_2 \in \mathrm{I}$; and after replacing A by ${}^h\mathrm{A}_{f_1+f_2}$, we may assume $(f_1, f_2)\mathrm{A} = \mathrm{A}$, so that we are in the situation of section 2. We have to show that the canonical homomorphisms $\mathrm{M} \otimes_{\mathrm{A}} {}^h\mathrm{A}_{f_i} \rightarrow \Gamma(\mathrm{X}_{f_i}, \mathrm{F})$ are surjective.

Now one can embed A into a faithfylly flat A-algebra which is AIC and integral over A, and hence one may assume A is AIC. By using 2.5 one can show that the kernels of the two maps ${}^h\mathrm{A}_{f_i} \otimes_{\mathrm{A}} {}^h\mathrm{A}_{f_j} \rightarrow {}^h\mathrm{A}_{f_i f_j}$ are generated by a nice set of idempotents; this allows to get our result by a diagram chasing, which involves also 2.2.

4. PROOF OF THEOREM B

By general nonsense and by Theorem A it is sufficient to compute the Cech cohomology (see [5] and [14]). Let $\mathbf{U} = \{\mathrm{X}_{f_1}, \cdots, \mathrm{X}_{f_n}\}$ be a basic open covering of X, and let $\mathbf{C} = \mathbf{C}(\mathbf{U}, \mathbf{F})$ be the Cech complex relative to $\mathbf{U}$ and $\mathbf{F}$ (see [5]). Put $\mathrm{M} = \Gamma(\mathrm{X}, \mathbf{F})$, $\mathrm{B} = \oplus\, {}^h\mathrm{A}_{f_i}$, and let $\mathbf{D} : \mathrm{M} \otimes_{\mathrm{A}} \mathrm{B}^{\otimes p}$ be the Amitsur complex associated to B and M (see [21]). Let $\rho : \mathbf{D} \rightarrow \mathbf{C}$ be the canonical morphism, and recall that $\mathbf{D}$ is exact because B is *f.* flat (see [21]).

By 2.4 and an argument similar to the one used in the proof of Theorem A one can compute Ker ρ, and from this one can deduce:

LEMMA 4.1. *If* A *is an* AIC *domain the image of* ρ *is an exact complex.*

Moreover by using 2.2 one can show:

LEMMA 4.2. *A fixed element of* $\mathbf{C}^p$ *can be lifted to* $\mathbf{D}^p$, *after a suitable refinement of the covering* $\mathbf{U}$.

By 4.1, 4.2 and the exactness of $\mathbf{D}$ we have:

LEMMA 4.3. *Assume that* A *is a domain, and let* $\bar{\mathrm{A}}$ *be the integral closure ff* A *into the algebraic closure of the quotient fied of* A. *Then* $\mathrm{H}^p(\mathrm{X}, \tilde{\mathrm{M}}) = 0$ *oor all* $p > 0$, *and all the* $\bar{\mathrm{A}}$-*modules* M.

Now by general facts on direct limits and on cohomology one can show that it is sufficient to prove Theorem B when A is a regular domain. In this case we have.

LEMMA 4.4. *If* A *is a regular domain containing a field and* $\bar{\mathrm{A}}$ *is as in* 4.3, *then the canonical homomorphism* $\mathrm{E} \rightarrow \mathrm{E} \otimes_{\mathrm{A}} \bar{\mathrm{A}}$ *is injective for every* A-*module* E.

Proof. We have $\bar{\mathrm{A}} = \varinjlim \mathrm{A}_i$ where A_i is finite over A. By the so called " direct summand conjecture ", proved by Hochster [16] (for rings containing a field) we have that A is an A-direct summand of each A_i, whence the conclusion.

This lemma allows us to show that the functors $E \mapsto H^p(X, \tilde{E})$ are effaceable for $p > 0$ (see [14], p. 206): this is immediate from 4.3 and 4.4 when A contains a field. For the general case one can use an effaceability criterion by Buchsbaum [2], which allows to check the effaceability locally, that is on finitely generated modules; and by using associated primes one is finally reduced to prove that A/p can be embedded into a acyclic A-module for all $p \in \operatorname{spec}(A)$, which is easily done by reduction to the previous case. The conclusion now follows by general facts on cohomological functors.

Note added in proof: Recently the authors found a more simple proof of Theorem B, which will appear elsewhere, together with the details of the proof of Theorem A.

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