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**On the existence of fixed points of holomorphic
automorphisms**

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Analisi matematica. — *On the existence of fixed points of holomorphic automorphisms.* Nota (*) di LÁSZLÓ L. STACHÓ, presentata dal Corrisp. G. CIMMINO.

RIASSUNTO. — Dato uno spazio di Banach complesso E , si considerano gli automorfismi ologomorfi del disco unità B di E , e si esamina il problema dell'esistenza di punti uniti nei casi in cui E sia lo spazio delle funzioni continue su un compatto od un reticolo di tipo M .

By a result of Kaup-Upmeyer [1], if E is any (complex) Banach space then every member of $\text{Aut } B(E)$ ($= \{\text{the Fréchet holomorphic automorphisms of the unit ball of } E\}$) can be considered as the restriction to $B(E)$ of a holomorphic map of some neighbourhood of $\bar{B}(E)$ ($= \text{the closed unit ball}$) into E . Several well-known positive examples suggested the following conjecture: These natural extensions always have a fixed point in $\bar{B}(E)$ (e.g. if E is a Hilbert space [2], if $E = L^p(X, m)$ for some measure space and $1 \leq p < \infty$ [3], if E is finite dimensional, if $E = l^\infty$). Our main purpose will be to investigate the validity of this conjecture and to prove that the answer is negative. The next observation immediately provides a large class of counter-examples. We shall denote by $\text{Aut } \bar{B}(E)$ the set formed by the continuous extensions to $\bar{B}(E)$ of the elements of $\text{Aut } B(E)$. Let Δ be the unit disc in \mathbf{C} ; the group $\text{Aut } \bar{\Delta}$ is the group of Moebius transformations.

THEOREM 1. *If $E = C(\Omega)$ for some compact space Ω then every $F \in \text{Aut } \bar{B}(E)$ admits a fixed point only if the following topological condition on Ω holds*

(\star) *for all open \mathcal{F}_σ -subsets G of Ω , any $g \in C_{\text{bounded}}(G)$ can be continuously extended to the whole Ω .*

Now it is naturally raised the question, for which kind of $F \in \text{Aut } \bar{B}(C(\Omega))$ condition (\star) ensures the existence of fixed points (in case of compact Ω). In this direction the following theorem holds:

THEOREM 2. *If Ω is a compact space then (\star) is equivalent to*

($\star\star$) *every $F \in \text{Aut } \bar{B}(C(\Omega))$ of the form $F(f) = [x \rightarrow M(x)f(Tx)]$ where T is any pointwise periodic homeomorphism of Ω onto itself (pointwise periodic meaning $\forall x \in \Omega \exists n \in \mathbf{N} T^n x = x$) and $M(\cdot)$ denotes any continuous map of Ω into $\text{Aut } \bar{\Delta}$, has a fixed point.*

The key lemma in the proof of Theorem 2 is the following one (which may have some interest also in itself).

(*) Pervenuta all'Accademia il 28 giugno 1979.

LEMMA 1. *If Ω is a compact space having the property (\star) and if $T : \Omega \leftrightarrow \Omega$ is any pointwise periodic homeomorphism, then, for every $f \in C(\Omega)$, there exists some $n \in \mathbf{N}$ such that $f = f \circ T^n$.*

Since, by a theorem of E. Vesentini [4], if Ω is any compact space and $F \in \text{Aut } \overline{B}(C(\Omega))$, then there exists a unique pair of continuous maps $M_F : \Omega \rightarrow \text{Aut } \overline{B}$ and $T_F : \Omega \leftrightarrow \Omega$ such that $F(f) = [x \rightarrow M_F(x) \cdot f(T_F x)]$ $\forall f \in \overline{B}(C(\Omega))$, one can conjecture that (\star) implies the existence of fixed points also for all $F \in \text{Aut } \overline{B}(C(\Omega))$. However, as it is seen from the next theorem, this expectation is false.

THEOREM 3. *One can construct a $\Phi \in \text{Aut } \overline{B}(L^\infty(0, 1))$ which has no fixed point.*

(The proof of this Theorem is essentially combinatorial, but requires also some ergodic theory).

COROLLARY. *Even the hyperstonianity (which is clearly a stronger property than (\star)) of a compact Ω does not ensure, in general, the existence of fixed points for every $F \in \text{Aut } \overline{B}(C(\Omega))$. (For the spectrum of $L^\infty(0, 1)$ is a compact hyperstonian space).*

On the other hand, Theorem 3 is suitable in proving the next positive result.

THEOREM 4. *If E is a complex M -space (for definition see [5]) with separable predual then every $F \in \text{Aut } \overline{B}(E)$ has fixed point iff $E \cong l^\infty$ or if E is finite dimensional.*

The proof of Theorem 4 is based on a characterization of M -spaces due to Rieffel-Kikutani [5], on Halmos's Isomorphism Theorem [6] and on

LEMMA 2. *If Ω is a discrete (not necessarily compact) space and $E = C_{\text{bounded}}(\Omega)$ then every $F \in \text{Aut } \overline{B}(E)$ can be written in the form $F(f) = [x \rightarrow M(x) \cdot f(Tx)]$ for suitable maps $M : \Omega \rightarrow \text{Aut } \overline{B}$ and $T : \Omega \leftrightarrow \Omega$.*

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