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Analytical theory of nonlinear oscillations X: some classes of equations $\ddot{x} + g(x) = 0$ with no finite Fourier series solution

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Equazioni differenziali ordinarie. — *Analytical theory of non-linear oscillations* X: *some classes of equations* $\ddot{x} + g(x) = 0$ *with no finite Fourier series solution.* Nota (*) di CHIKE OBI, presentata dal Socio G. SANSONE.

RIASSUNTO. — Si dimostra che l'equazione considerata nel titolo ammette soluzioni periodiche rappresentabili da una serie di Fourier con un numero finito di termini solo se $g(x)$ è lineare.

§ 1. Let us say that a solution $x(t)$ of the differential equation

$$\ddot{x} + g(x) = 0 \qquad (\dot{x} = dx/dt) \tag{1.1}$$

in the real domain is a *finite Fourier series solution* of degree n if $x(t)$ has the least period $2\pi\rho^{-1}$ in t and its Fourier series in the interval $0 \leq t \leq 2\pi\rho^{-1}$ terminates and is of the form

$$x(t) = \frac{1}{2} A_0 + \sum_{r=1}^{n} (A_r \cos r\rho t + B_r \sin x\rho t)$$

where $(A_n, B_n) \neq (0, 0)$.

In papers VII, VIII and IX [1, 2, 3] of this series we asserted that when $g(x)$ is non-linear and every solution of (1.1) oscillates with the same least period then equation (1.1) has no finite Fourier series solution. The sole aim of this paper is to give a proof of this assertion formalised in Theorem 2 below. In the process of achieving this aim we prove Theorem 1 below which establishes some classes of equations of the form (1.1) with no finite Fourier series solutions. The paper is an extract from an unpublished paper on equations of the form (1.1) with finite Fourier series solutions referred to in papers VII, VIII and IX of this series.

It is well-known that

(1) If there is an interval $|x| < A$ in which (i) $g(x)$ is continuous, (ii) $\operatorname{sgn} g(x) = \operatorname{sgn} x$, and (iii) a solution of (1.1) with an initial condition in $|x| < A$ is unique then the solution of (1.1) with a stationary value $\alpha > 0$ at $t = 0$ is an even periodic function of t for all α in a certain interval $0 < \alpha < A_1 < A$.

Let $\phi_0 = \phi_0(t) = \phi_0(\alpha, \rho t)$ denote the solution just mentioned and let $2\pi\rho^{-1} = 2\pi\rho^{-1}(\alpha)$ denote its least period. The basic question of which this paper is a partial answer is this: Is there a set of values α_0 of α in $0 < \alpha < A_1$

(*) Pervenuta all'Accademia il 21 agosto 1979.

or in $0 < \alpha < \infty$ such that $\phi_0(\alpha, \rho t)$ is a finite Fourier series whenever $\alpha = \alpha_0$? Our interest in this question (which can stand on its own as a purely analytical question) arises from well-known results in the theory of non-linear oscillations which indicate that the number of frequencies of a periodic oscillation of a perturbation of (1.1) depends on the number of frequencies of $\phi_0(\alpha, \rho t)$, and when the latter is infinite so is the former.

§ 2. Since ϕ_0 is even in t it follows that its Fourier series (in its interval of periodicity $0 \leq t \leq 2\pi\rho^{-1}$) is of the form

$$\phi_0 = a(\alpha; 0) + \sum_{r=1} a(\alpha; m_r) \cos m_r \rho(\alpha) t$$

where none of the coefficients $a(\alpha; m_r)$ $(r = 1, 2, \cdots)$ is identically 0, and where the sequence (m_r) $(r = 1, 2, \cdots)$ is a strictly ascending sequence of positive integers. It is easy to see that if ϕ_0 is a finite Fourier series of degree n, then ϕ_0 can be put in the form

$$\phi_0 = P_n(u), \qquad u = \cos \rho t, \tag{2.1}$$

where

$$P_n(u) = \sum_{r=0}^{n} a_r u^r, \qquad a_n \neq 0, \tag{2.2}$$

and the coefficients $a_r = a_r(\alpha)$ are independent of u.

The change of variables from t to u transforms equation (1.1) to

$$\rho^2 (1 - u^2) x^{11} - \rho^2 u x^1 + g(x) = 0 \tag{2.3}$$

where $x^1 = dx/du$.

Since ϕ_0 satisfies (1.1) it follows from (2.3) that

$$g(\phi_0) = \rho^2 u \phi_0^1 - \rho^2 (1 - u^2) \phi_0^{11}$$

and so it follows from (2.1) that

$$g(\phi_0) = Q_n(u) \tag{2.4}$$

where

$$Q_n(u) = \sum_{r=0}^{n} b_r u^r, \qquad b_n = \rho^2 n^2 a_n. \tag{2.5}$$

The following Theorem is imminent.

THEOREM 1. *If subject to* (i), (ii) *and* (iii) *of* (1°) *of* § 1 *equation* (1.1) *has a finite Fourier series solution of degree n and least period $2\pi\rho^{-1}$ then there exists a constant $\mu > 0$ such that*

(a) $x^{-1} g(x) \to \mu^2$ *as* $|x| \to \infty$ *if n is odd*;

(b) *if n is even then either $g(x)$ is complex as $x \to -\infty$, and $x^{-1}g(x) \to \mu^2$ as $x \to \infty$ or $g(x)$ is complex as $x \to \infty$, and $x^{-1}g(x) \to \mu^2$ as $x \to -\infty$;*

(c) $\rho n = \mu$.

For, if ϕ_0 is a finite Fourier series of degree n then by (2.1) and (2.4) we have

$$x = \mathrm{P}_n(u), \qquad g(x) = \mathrm{Q}_n(u) \tag{2.6}$$

which with (2.2) and (2.5) give

$$xu^{-n} \to a_n \neq 0, \qquad g(x)\, u^{-n} \to a_n\, \rho^2\, n^2$$

as $|u| \to \infty$.

§ 3. THEOREM 2. *If subject to* (i), (ii) *and* (iii) *of* (1°) *of* § 1 *all the solutions of equation* (1.1) *oscillate with the same least period then equation* (1.1) *has a finite Fourier series solution if and only if $g(x)$ is linear.*

Proof. Suppose that all the hypotheses hold. The hypothesis that all the solutions of (1.1) oscillate with the same least period implies, inter alia, that $\operatorname{sgn} g(x) = \operatorname{sgn} x$ in $|x| < \infty$ and so implies by Theorem 1 (b) that every finite Fourier series solution of (1.1) is of odd degree, and by Theorem 1 (a) that there is a constant $\mu > 0$ such that $x^{-1} g(x) \to \mu^2$ as $|x| \to \infty$. Since $\operatorname{sgn} g(x) = \operatorname{sgn} x$ in $|x| < \infty$ we have for the least period $2\pi\rho^{-1}$ of $\phi_0(\alpha, \rho t)$ (defined in § 1) the formula

$$\pi\rho^{-1} = \int_a^\alpha \{2\,\mathrm{G}(\alpha) - 2\,\mathrm{G}(\xi)\}^{-1/2}\, d\xi \tag{3.1}$$

where

$$\mathrm{G}(a) = \mathrm{G}(\alpha), \qquad a\alpha < 0 \quad \text{and} \quad \mathrm{G}(x) = \int_0^x g(\xi)\, d\xi. \tag{3.2}$$

If ρ is independet of α then

$$\pi\rho^{-1} = \lim_{a \to \infty} \int_a^\alpha \{2\,\mathrm{G}(\alpha) - 2\,\mathrm{G}(\xi)\}^{-1/2}\, d\xi.$$

Hence, since $x^{-1} g(x) \to \mu^2$ as $|x| \to \infty$ so that $a\alpha^{-1} \to -1$ as $\alpha \to \infty$, we have

$$\pi\rho^{-1} = u^{-1} \int_{-1}^{1} (1 - \xi^2)^{-1/2}\, d\xi,$$

that is to say, $\rho = \mu$. Substituting $\rho = u$ in the equation $\rho n = \mu$ of Theorem 1 (c) we get $n = 1$. Hence in this case we have (cfr. (2.6))

$$x = a_0 + a_1 \cos \mu t$$

$$g(x) = \mu^2 a_1 \cos \mu t .$$

So $g(x) = \mu^2 (x - a_0)$ and the theorem is established.

References

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[2] C. Obi (1978) - *Analytical theory of non-linear oscillations* VIII, « Ann. Mat. Pura Appl. », IV Vol. CXVII, 339-347.
[3] C. Obi (1979) - *Analytical theory of non-linear oscillations* IX, « Ann. Mat. Pura Appl. », IV Vol. CXX, 139-157.