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Some remarks on oscillation of second order differential equations

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Equazioni differenziali ordinarie. — *Some remarks on oscillation of second order differential equations* (*). Nota (**) di LU-SAN CHEN, presentata dal Socio G. SANSONE.

RIASSUNTO. — Si espongono alcuni risultati sulle proprietà oscillatorie delle soluzioni di alcune equazioni non lineari del secondo ordine.

1. INTRODUCTION

There are many results on oscillatory property of solutions of differential equations. In this paper, we shall discuss oscillatory property of solutions of second order differential equations of the form

$$(r(t)\,x'(t))' + f(t, x, x') = 0, \tag{1}$$

and more general functional differential equations of the form

$$(r(t)\,x'(t))' + f(t, x(\delta_1(t)), x'(\delta_2(t))) = 0, \tag{2}$$

where $r(t) > 0$ is continuous on $I = (0, \infty)$ and $f(t, x, u)$ is continuous on $I \times R \times R$, $R = (-\infty, \infty)$ with the following conditions:

(i) *for* $t \geq 0$ *and* $x(t) \geq 0$, *there exist continuous functions* $a(t)$ *and* $\alpha(x)$ *such that*

$$\liminf_{t\to\infty} \int_T^t a(s)\,ds \geq 0 \qquad \textit{for all large } T$$

and that $x\alpha(x) > 0\ (x \neq 0)$, $\alpha'(x) \geq 0$ *and for all large* t, $x \geq 0$, $|u| < \infty$

$$a(t)\,\alpha(x) \leq f(t, x, u),$$

(ii) *for* $t \geq 0$ *and* $x(t) \leq 0$, *there exist continuous functions* $b(t)$ *and* $\beta(x)$ *such that*

$$\liminf_{t\to\infty} \int_T^t b(s)\,ds \geq 0 \qquad \textit{for all large } T$$

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and that $x\beta(x) > 0\ (x \neq 0)$, $\beta'(x) \geq 0$ *and for all large* $t, x \leq 0, |u| < \infty$

$$f(t, x, u) \leq b(t)\, \beta(x).$$

And $\delta_i(t)\ (i = 1, 2)$ *are continuous on* I *and* $\lim_{t\to\infty} \delta_i(t) = \infty\ (i = 1, 2)$.

We consider only extendable solutions of (2). A solution $x(t)$ of (2) is said to be *oscillatory* if it has arbitrarily large zeros, and (2) is said to be *oscillatory* if all its solutions are oscillatory. For the special case

$$x''(t) + a(t) f(x) = 0, \tag{1'}$$

and

$$x''(t) + a(t) f(x(\delta(t))) = 0, \tag{2'}$$

there exist numerous results.

In the present paper, we shall extend some results which have been obtained for (1′) and (2′) to the more general equations (1) and (2) for the case $\int^{\infty} \frac{dt}{r(t)} = \infty$ respectively.

2. Main results

THEOREM 1. *Let* $f(t, x, u)$ *satisfy the conditions* (i) *and* (ii), *and assume that*

$$\int_0^{\infty} a(t)\, dt < \infty, \int_0^{\infty} \left(\frac{1}{r(s)} \int_s^{\infty} a(u)\, du\right) ds = \infty,$$

$$\int_0^{\infty} b(t)\, dt < \infty, \int_0^{\infty} \left(\frac{1}{r(s)} \int_s^{\infty} b(u)\, du\right) ds = \infty.$$

Let

$$\int_{\varepsilon}^{\infty} \frac{du}{\alpha(u)} < \infty, \int_{-\varepsilon}^{-\infty} \frac{du}{\beta(u)} < \infty,$$

for some $\varepsilon > 0$. *Then equation* (1) *is oscillatory.*

Proof. Suppose $x(t)$ is nonoscillatory. Then $x(t)$ eventually attains a constant sign. Without any loss suppose there exists a τ_1 such that for $t > \tau_1$, $x(t) > 0$. (the case when $x(t) < 0$ can be handled similarly).

From (1) and (i), we have

$$\frac{(r(t)\, x'(t))'}{\alpha(x(t))} \leq -a(t). \tag{3}$$

Integrating (3) from sufficiently large $c\ (\geq q)$ to $t \geq c$, we obtain

$$\int_c^t \frac{(r(s)\,x'(s))'}{\alpha(x(s))}\,ds \leq -\int_c^t a(s)\,ds, \tag{4}$$

then, integrating by parts, we get

$$\left[\frac{r(s)\,x'(s)}{\alpha(x(s))}\right]_c^t - \int_c^t \frac{r(s)\,x'(s)\,[-\alpha'(x(s))\,x'(s)]}{[\alpha(x(s))]^2}\,ds \leq -\int_c^t a(s)\,ds,$$

so that

$$\frac{r(t)\,x'(t)}{\alpha(x(t))} \leq \frac{r(c)\,x'(c)}{\alpha(x(c))} - \int_c^t a(s)\,ds. \tag{5}$$

Let $x'(t) \geq 0$ for all large $t \geq d$. For $t \geq d$ (5) gives

$$\int_t^\infty a(s)\,ds \leq \frac{r(t)\,x'(t)}{\alpha(x(t))},$$

may be written as

$$\frac{1}{r(t)}\int_t^\infty a(s)\,ds \leq \frac{x'(t)}{\alpha(x(t))}. \tag{6}$$

Integration of (6) from d to $t \geq d$, yields

$$\int_d^t \left(\frac{1}{r(s)}\int_s^\infty a(u)\,du\right) ds \leq \int_d^t \frac{x'(s)}{\alpha(x(s))}\,ds = \int_{x(d)}^{x(t)} \frac{dx}{\alpha(x)} < \infty.$$

This is a contradiction.

If $x'(t)$ oscillates, we can assume that there exists a sequence $\{c_n\}_{n=1}^\infty$ with the following properties

(I) $c_n \to \infty$ as $n \to \infty$,

(II) $x'(c_n) < 0$, $n = 1, 2, \cdots$,

(III) Take c_n so large that $\int_t^\infty a(s)\,ds \geq 0$ for $t \geq c_n > q$.

Then by (5)

$$\frac{r(t)\,x'(t)}{\alpha(x(t))} \leq \frac{r(c_n)\,x'(c_n)}{\alpha(x(c_n))} - \int_{c_n}^t a(s)\,ds$$

and

$$\int_{c_n}^{t} a(s)\,ds \geq 0,$$

for sufficiently large $t \geq T \geq c_n$. Hence we conclude that

$$(7) \qquad x'(t) < 0 \qquad \text{for all} \quad t \geq T.$$

In this case, we have

$$\int_T^t a(s)\,\alpha(x(s))\,ds = \alpha(x(t))\int_T^t a(s)\,ds - \int_T^t \alpha'(x(s))\,x'(s)\left[\int_T^s a(u)\,du\right]ds \geq 0$$

for sufficiently large $t \geq k \geq T$. By (1)

$$(8) \qquad (r(t)\,x'(t))' = -f(t, x, x') \leq -a(t)\,\alpha(x(t)).$$

Integrating (8) from T to $t \geq T$, we have

$$r(t)\,x'(t) - r(T)\,x'(T) \leq -\int_T^t a(s)\,\alpha(x(s))\,ds \leq 0$$

for all $t \geq k$, so that

$$x'(t) \leq \frac{r(T)\,x'(T)}{r(t)} < 0$$

for all $t \geq k$, which implies $x(t) < 0$ for all sufficiently large t. This is a contradiction.

THEOREM 2. *Let $f(t, x, u)$ satisfy the conditions* (i), (ii) *and assume that*

$$\int_0^\infty a(t)\,dt = \infty, \quad \int_0^\infty b(t)\,dt = \infty.$$

The equation (1) *is oscillatory.*

Proof. From (5), we have at once

$$\frac{r(t)\,x'(t)}{\alpha(x(t))} \to -\infty \qquad \text{as} \quad t \to \infty.$$

Hence, we obtain $x'(t) < 0$ for sufficiently large t. Thus, we have at once a contradiction as in the proof of case 2 of Theorem 1.

THEOREM 3. *Let $f(t, x, u)$ satisfy the conditions* (i), (ii) *and assume that*

$$(9) \qquad \int_y^\infty \frac{sa(s)}{r(s)}\,ds = \infty, \int_y^\infty \frac{sb(s)}{r(s)}\,ds = \infty.$$

Let

$$\int_{\varepsilon}^{\infty} \frac{\mathrm{d}u}{\alpha(u)} < \infty \qquad \text{and} \qquad \int_{-\varepsilon}^{-\infty} \frac{\mathrm{d}u}{\beta(u)} < \infty \quad .$$

for some $\varepsilon > 0$. *Then equation* (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution. Without loss of generality assume $x(t) \neq 0$ on $[a^*, \infty)$ for some $a^* > 0$. Suppose $x(t) > 0$ (The case $x(t) < 0$ is handled similarly). Then from (1) we obtain

$$\frac{t(r(t)x'(t))'}{r(t)\,\alpha(x(t))} \leq -\frac{ta(t)}{r(t)} \,. \tag{10}$$

Integrating (10) from a^* to $t \geq a^*$ gives

$$\frac{tx'(t)}{\alpha(x(t))} \leq \int_{x(a^*)}^{x(t)} \frac{\mathrm{d}s}{\alpha(s)} - \int_{a^*}^{t} \frac{sa(s)}{r(s)}\,\mathrm{d}s - \int_{a^*}^{t} \frac{sx'(s)\,r'(s)}{r(s)\,\alpha(x(s))}\,\mathrm{d}s + \frac{a^*x'(a^*)}{\alpha(x(a^*))} \,,$$

which implies that

$$\frac{tx'(t)}{\alpha(x(t))} \to -\infty \qquad \text{as} \quad t \to \infty \,,$$

so that we have

$$x'(t) < 0 \qquad \text{for all sufficiently large } t. \tag{11}$$

By (i), we may suppose that T_1 is sufficiently large, so that $\int_{T_1}^{t} a(s)\,\mathrm{d}s \geq 0$ and $x'(t) < 0$ for all $t \geq k \geq T_1$. Hence we have

$$\int_{T_1}^{t} a(s)\,\alpha(x(s))\,\mathrm{d}s = \alpha(x(t)) \int_{T_1}^{t} a(s)\,\mathrm{d}s - \int_{T_1}^{t} \alpha'(x(s))\,x'(s) \left[\int_{T_1}^{s} a(u)\,\mathrm{d}u\right] \mathrm{d}s \geq 0 \,, \tag{12}$$

$t \geq k$. From (1) it follows that

$$r(t)\,x'(t) - r(T_1)\,x'(T_1) + \int_{T_1}^{t} a(s)\,\alpha(x(s))\,\mathrm{d}s \leq 0 \,,$$

so that $r(t)\,x'(t) \leq r(T_1)\,x'(T_1) < 0$, $t \geq k$ which implies $x(t) < 0$ for all large t. This contradicts $x(t) > 0$.

Let us consider the functional equation of (2).

THEOREM 4. *Let* $f(t, x, u)$ *satisfy the conditions* (i), (ii) *and assume that* $\int_0^{\infty} a(s)\,\mathrm{d}s = \infty$ *and* $\int_0^{\infty} b(s)\,\mathrm{d}s = \infty$ *and* $\delta_1'(t) \geq 0$. *Then* $x'(t)$ *is oscillatory for any solution* $x(t)$ *of* (2).

Proof. Suppose $x'(t)$ is not oscillatory. If $x(t)$ oscillates, then $x'(t)$ oscillates, which leads to a contradiction. Hence, we suppose $x(t)$ is ultimately positive (the case $x(t) < 0$ can be treated similarly). Then so is $x(\delta_1(t))$ and $\alpha(x(\delta_1(t))$.

Let

$$W(t) = \frac{r(t)\, x'(t)}{\alpha(x(\delta_1(t))} \,.$$

Then we have

$$W'(t) \leq -a(t) - \frac{r(t)\, x'(t)\, x'(\delta_1(t))\, \alpha'(x(\delta_1(t)))\, \delta_1'(t)}{[\alpha(x(\delta_1(t)))]^2} \,.$$

Since $x'(t)$ is not oscillatory, we have $x'(t)\, x'(\delta_1(t)) > 0$ for sufficiently large t. Thus $W'(y) \leq -a(t)$. Integrating from c to $y \geq c$, we obtain

$$W(y) \leq W(c) - \int_c^y a(t)\, dt \,.$$

By assumption, we have

$$x'(t) < 0 \qquad \text{for all sufficiently large } t. \tag{13}$$

Consequently,

$$\int_T^t a(s)\, \alpha(x(\delta_1(s)))\, ds = \alpha(x(\delta_1(t))) \int_T^t a(s)\, ds$$

$$- \int_T^t \alpha'(x(\delta_1(s)))\, x'(\delta_1(s))\, \delta_1'(s) \left[\int_T^s a(u)\, du\right] ds \geq 0 \,.$$

From (2), we get

$$(r(t)\, x'(t))' \leq -a(t)\, \alpha(x(\delta_1(t))) \,.$$

Integrating we obtain $x'(t) < 0$. Hence $x(t) < 0$ for sufficiently large t. Thus contradicts the fact $x(t) > 0$. We conclude that $x'(t)$ is oscillatory.

THEOREM 5. *Let $f(t, x, u)$ satisfy the conditions* (i), (ii) *and assume that*

$$\int_0^\infty a(t)\, dt = \infty \,, \quad \int_0^\infty b(t)\, dt = \infty \,.$$

Let $\delta_1(t)$ be defferentiable with $0 \leq \delta_1'(t) \leq 1$. Finally assume that $a(t) \geq 0$. Then equation (2) *is oscillatory.*

Proof. Suppose $x(t)$ is ultimately positive (the case $x(t) < 0$ can be treated similarly). Then from (2)

$$(r(t)\,x'(t))' = -f(t\,,x(\delta_1(t))\,,x'(\delta_2(t))) \leq -a(t)\,\alpha(x(\delta_1(t))) \leq 0,$$

which implies that $r(t)\,x'(t)$ is nonincreasing. From the assumption it follows that $x'(t) > 0$ for all sufficiently large t. Let

$$W(t) = \frac{r(t)\,x'(t)}{\alpha(x(\delta_1(t)))}\,.$$

Then $W'(t) \leq -a(t)$. Integrating it, we obtain

$$W(y) \leq W(c) - \int_c^y a(t)\,dt\,. \tag{14}$$

From the assumption, it follows that $x'(t) < 0$ for all sufficiently large t. This is a contradiction. Hence theorem is proved.

THEOREM 6. *Let $f(t\,,x\,,u)$ satisfy the conditions* (i), (ii) *and assume that*

$$\int_0^\infty a(t)\,dt < \infty\;,\quad \int_0^\infty\left(\frac{1}{r(s)}\int_s^\infty a(u)\,du\right)ds = \infty\,,$$

$$\int_0^\infty b(t)\,dt < \infty\;,\quad \int_0^\infty\left(\frac{1}{r(s)}\int_s^\infty b(u)\,du\right)ds = \infty\,.$$

In addition to the assumptions of Theorem 5, assume that $\delta_i(t) = t - \tau_i(t)$, $0 < \tau_i(t) \leq M$ $(i = 1\,,2)$, where $\tau_i(t)$ is continuous and M is positive constant. Let

$$\int_\varepsilon^\infty \frac{du}{\alpha(u)} < \infty\;,\quad \int_{-\varepsilon}^{-\infty}\frac{du}{\beta(u)} < \infty$$

for some $\varepsilon > 0$. Then equation (2) is oscillatory.

Proof. By [7],

$$\left|\frac{x(t-\tau_1(t))}{x(t)} - 1\right| = \frac{x(t) - x(t-\tau_1(t))}{x(t)} \leq \frac{x(t) - x(t-M)}{x(t)}$$

and by the mean-value theorem,

$$\frac{x(t) - x(t-M)}{x(t)} = \frac{Mx'(\xi)}{x(t)} \leq M\,\frac{x'(t-M)}{x(t)} \leq M\,\frac{x'(t-M)}{x(t-M)}\,.$$

Thus, we obtain

$$\left|\frac{x(t-\tau_1(t))}{x(t)} - 1\right| < M\,\frac{x'(t-M)}{x(t-M)} \tag{15}$$

for $t \in [t_0 + \mathrm{M}, \infty)$. After a simple computation we have

$$\lim_{t\to\infty} \frac{x'(t)}{x(t)} = 0 .$$

from (15) we obtain

$$\lim_{t\to\infty} \frac{x(\delta_1(t))}{x(t)} = 1 . \tag{16}$$

From (16) we get

$$\frac{x(\delta_1(t))}{x(t)} > \tfrac{1}{2} \tag{17}$$

for $t \geq \mathrm{T}$ (sufficiently large). In view of assumption, then (14) and (17) give

$$\frac{1}{r(t)} \int_t^\infty a(s)\,\mathrm{d}s \leq \frac{x'(t)}{\alpha(x(\delta_1(t)))} \leq \frac{x'(t)}{\alpha(\frac{1}{2}x(t))}$$

integrating it from d to $t \geq d$, we get

$$\int_d^t \left(\frac{1}{r(t)} \int_s^\infty a(u)\,\mathrm{d}u\right) \mathrm{d}s \leq \int_d^t \frac{x'(s)}{\alpha(\frac{1}{2}x(s))}\,\mathrm{d}s = 2 \int_{\frac{1}{2}x(d)}^{\frac{1}{2}x(t)} \frac{\mathrm{d}s}{\alpha(s)} < \infty .$$

which leads to a contradiction. This proves the theorem.

Remark. For the case $r(t) = 1$, Theorems 1 and 2 are generalizes some result of Coles [3], Theorem 3 is a generalization of results of Kartsatos [4], Onose [5], [6], Theorem 4 is an extends result of Travis [8], Theorems 5 and 6 are generalizes a results of Burkowski [2].

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