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Asymptotic Behavior of Resolvents in Banach Spaces

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Analisi funzionale. — *Asymptotic Behavior of Resolvents in Banach Spaces.* Nota (*) di SIMEON REICH, presentata dal Socio G. SANSONE.

RIASSUNTO. — Si studia il comportamento asintotico di risolventi di operatori accretivi negli spazi di Banach.

Let E be a Banach space, $A \subset E \times E$ an accretive operator with domain D (A) and range R (A), and $J_t = (I + tA)^{-1}$ the resolvent of A. The purpose of this paper is to study the asymptotic behavior of J_t as $t \to \infty$, and to compare it with the asymptotic behavior of the semigroup S generated by — A. We denote the closure of a subset D of E by *cl* (D), and refer the reader to [5] for terms not defined here.

THEOREM 1. *Suppose that* $A \subset E \times E$ *is accretive, cl* (D (A)) *is a nonexpansive retract of* E, *and* $R(I + rA) \supset cl(D(A))$ *for all* $r > 0$. *If the norm of* E *is uniformly Gâteaux differentiable and the norm of* E* *is Fréchet differentiable, then for each x in cl* (D (A)), *the strong* $\lim_{t\to\infty} J_t x/t = -v$, *where v is the element of least norm in cl* (R (A)).

Proof. We first show that in an arbitrary Banach space, $\lim_{t\to\infty} |J_t x/t| = d(0, R(A))$. Indeed, denoting $d(0, R(A))$ by d and the Yosida approximation of A by A_t, we have, on the one hand, $\liminf_{t\to\infty} |A_t x| \geq d$ because $A_t x \in R(A)$. On the other hand, given $\varepsilon > 0$, there is $[y, z] \in A$ such that $|z| \leq d + \varepsilon$. Since $|A_t x| \leq |A_t x - A_t y| + |A_t y| \leq 2|x - y|/t + |z|$, $\limsup_{t\to\infty} |A_t x| \leq d$, and the result follows. By [12, Theorem 4.1], *cl* (D (A)) is, in fact, a sunny nonexpansive retract of E. Consequently, *cl* (R (A)) has the minimum property by [7, Theorem 1]. Since E* has a Fréchet differentiable norm, it now follows that $\lim_{t\to\infty} A_t x = v$.

Theorem 1 extends results of Morosanu [2] and Pazy [3] for monotone operators in Hilbert space. Their methods are different from ours. It improves upon [12, Proposition 5.2] because we no longer assume that E has a Fréchet differentiable norm. It certainly applies to all L^p spaces, $1 < p < \infty$, but it does not hold for all Banach spaces: $E = l^1$ and $Ax = (x_1 - 1, x_2 - x_1, x_3 - x_2, \cdots)$ provide us with a counter-example. It is completely analogous

(*) Pervenuta all'Accademia il 2 luglio 1979.

to the known result for semigroups [7, Theorem 2]. In our case, however, we also have $\lim_{t\to\infty} A^0 J_t x = v$, while the analogous result is not true for all semigroups. Another difference between resolvents and semigroups is brought out by our next result.

THEOREM 2. *Let* E *be a Banach space each bounded closed convex subset of which has the fixed point property for nonexpansive mappings.* Let $A \subset E \times E$ *be accretive with* $R(I + rA) \supset cl(D(A))$ *for all* $r > 0$, *and assume that* $cl\, D(A))$ *is convex. Then* A *is zero free if and only if* $\lim_{t\to\infty} |J_t x| = \infty$ *for each* x *in* $cl(D(A))$.

Proof. If $y \in A^{-1} 0$, then $|A_t x| \leq 2 |x - y|/t$, so that $|J_t x|$ is bounded. Conversely, if $\{x_n = J_{t_n} x\}$ is bounded for some x in $cl(D(A))$ and for some sequence $t_n \to \infty$, then $|x_n - J_r x_n| \leq r \| Ax_n \| \leq r |x - x_n|/t_n \to 0$. Pick a point y in $cl(D(A))$, and set $R = \limsup_{n\to\infty} |y - x_n|$. The set $\{z \in cl(D(A)) : \limsup_{n\to\infty} |z - x_n| \leq R\}$ is non-empty, bounded, closed, convex, and invariant under J_r (cf. [4, Theorem 1]). Therefore it contains a fixed point of J_r, which is a zero of A.

Note that if A is m-accretive, there is no need to assume that $cl(D(A))$ is convex (although this is true when E^* has a Fréchet differentiable norm).

If indeed $0 \in R(A)$, we have the following convergence result (cf. [5, Theorem 5.1] and [9, Proposition 4]).

THEOREM 3. *Let* E *be a smooth uniformly convex Banach space with a duality map that is weakly sequentially continuous at* 0, *and let* $A \subset E \times E$ *be an accretive operator such that* $cl(D(A))$ *is convex and* $R(I + rA) \supset cl(D(A))$ *for all* $r > 0$. *If* $0 \in R(A)$, *then the strong* $\lim_{t\to\infty} J_t x = Qx$ *for each* x *in* $cl(D(A))$ *where* Q *is the unique sunny nonexpansive retraction of* $cl(D(A))$ *onto* $A^{-1} 0$.

Again, the analogous result does not hold for all semigroups. Theorem 3 has already been used in the study of certain explicit and implicit iterative methods. See, for example, [11, Theorem 4] and [9, Theorem 1]. These methods are defined by

$$x_{n+1} \in x_n - h_n (Ax_n + p_n x_n) \tag{1}$$

and

$$x_n \in x_{n+1} + h_{n+1} (Ax_{n+1} + p_{n+1} x_{n+1}) \tag{2}$$

respectively. Theorems 1 and 2 imply that if $0 \notin R(A)$, then in both cases $|x_n| \to \infty$, and $p_n x_n \to -v$, where v is the element of least norm in $cl(R(A))$. In contrast with Theorem 1 and 2, we do not know if Theorem 3 is valid for L^p, $1 < p < \infty$, $p \neq 2$. (It is valid, of course, for l^p, $1 < p < \infty$; in fact, all smooth Orlicz sequence spaces have duality maps that are weakly sequentially continuous at 0).

Another aspect of the asymptotic behavior of resolvents is obtained when one considers the sequence defined inductively by

$$x_n = J_{t_n} x_{n-1}, \tag{3}$$

where $\{t_n\}$ is a positive sequence. The following result is achieved by combining [10, Theorem 3] with [8, Theorems 2 and 3].

THEOREM 4. *Let* A *be an m-accretive operator in a Banach space* E, *and suppose that both* E *and* E* *are uniformly comvex. Assume that the modulus of convexity of* E *satisfies* $\delta(\varepsilon) \geq C\varepsilon^r$ *for some* $r \geq 2$ *and* $C > 0$, *and that* $\sum_{n=1}^{\infty} t_n^r = \infty$. *If* $\{x_n\}$ *is defined by* (3), *and* v *is the element of least norm in* $cl(R(A))$, *then*

(a) $\lim_{n\to\infty} A^0 x_n = v$;

(b) $0 \notin R(A)$ *if and only if* $\lim_{n\to\infty} |x_n| = \infty$;

(c) *If* $0 \in R(A)$, *then the weak* $\lim_{n\to\infty} x_n$ *exists and belongs to* $A^{-1}0$.

Theorem 4 can be applied to L^p, $1 < p < \infty$. In contrast with Theorem 3, we do not have strong convergence in (c) in general, even in Hilbert space. In certain cases, another description of the limit in (c) is possible.

THEOREM 5. *Let* A *be an m-accretive operator in a smooth uniformly convex Banach space* E *with a duality map that is weakly sequentially continuous at* 0, *and let* $\{x_n\}$ *be defined by* (3). *Assume that* $0 \in R(A)$, *and let* $P : E \to A^{-1}0$ *be the nearest point projection. In the setting of Theorem* 4, *the weak* $\lim_{n\to\infty} x_n =$ *the strong* $\lim_{n\to\infty} Px_n$.

Proof. On the one hand, the weak $\lim_{n\to\infty} x_n$ is the asymptotic center of $\{x_n\}$ because E satisfies Opial's condition. On the other hand, the proof of [6, Proposition 2.1] can be used to show that in any uniformly convex space, the strong $\lim_{n\to\infty} Px_n$ exists and equals the asymptotic center of $\{x_n\}$. The result follows. We also see that in the present case there is no need to assume that E* is uniformly convex.

Returning to J_t and $S(t)$, we let E and E* be uniformly convex, $A \subset E \times E$ an m-accretive operator, and S the semigroup generated by $-A$. By Theorem 1, $\lim_{t\to\infty} |x - S(t)x| / |x - J_t x| = 1$ if $0 \notin cl(R(A))$ and $x \in cl(D(A))$. If $0 \in R(A)$, $x \notin A^{-1}0$, $t_n \to \infty$, and $J_{t_n}x$ converges weakly to z, then $z \in A^{-1}0$ and $|x - z| \leq \liminf_{n\to\infty} |x - J_{t_n}x|$. Also, $|x - S(t_n)x| \leq 2|x - z|$. It follows that $\limsup_{t\to\infty} |x - S(t)x| / |x - J_t x| \leq 2$. In Hilbert space this inequality is due to Pazy [3] who used a different argument.

He also showed that 2 is the best possible constant. (We always have $|x - S(t)x| \leq 3|x - J_t x|$). If S satisfies the conclusion of [1, Theorem 4.3] (equivalently, $\lim_{t\to\infty} A^0 S(t)x = v$ for all x in D (A)), then we also have, in case $A^{-1}0 \neq \emptyset$, $\liminf_{t\to\infty} |x - S(t)x| / |x - J_t x| \geq \frac{1}{2}$. In addition, the weak $\lim_{t\to\infty} S(t)x$ exists in this case and belongs to $A^{-1}0$. Also, if $A^{-1}0 = \emptyset$, then $|S(t)x| \to \infty$ as $t \to \infty$ for each x in cl(D (A)).

Added in proof: We have recently shown that Theorem 1 remains true even if cl(D (A)) is not a nonexpansive retract of E, and that Theorem 3 is valid for all uniformly smooth Banach spaces (hence, in particular, for all L^p spaces, $1 < p < \infty$). For more details, see our announcement entitled "A solution to a problem on the asymptotic behavior of nonexpansive mappings and semigroups", Abstracts Amer. Math. Soc., and our paper entitled "Strong convergence theorems for resolvents of accretive operators in Banach spaces", J. Math. Anal. Appl., to appear.

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