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The radius of convexity and starlikeness for certain classes of analytic functions with negative coefficients II

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Funzioni di variabile complessa. — *The radius of convexity and starlikeness for certain classes of analytic functions with negative coefficients II.* Nota (*) di SANGAPPA M. SARANGI (**) e BASAPPA A. URALEGADDI (**), presentata dal Socio G. SANSONE.

RIASSUNTO. — Vengono considerate particolari classi di funzioni analitiche e, sotto opportune ipotesi, vengono determinati i raggi dei campi ove esse sono convesse o di tipo stellato.

1. INTRODUCTION

Let $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ be analytic for $|z| < 1$ and $g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n$ be analytic and univalent on $\{z : |z| < 1\}$. In [3] we have determined coefficient estimates and comparable results for functions $f(z)$ which satisfy $\mathrm{Re}(f(z)/z) > \alpha$ and $\mathrm{Re} f'(z) > \alpha$ for $|z| < 1$. Also we have obtained the radius of convexity for functions $f(z)$ which satisfy $\mathrm{Re} f'(z) > \alpha$ for $|z| < 1$, and the radius of starlikeness for functions $f(z)$, which satisfy $\mathrm{Re}(f(z)/z) > \alpha$ for $|z| < 1$. Further we have determined the radius of starlikeness for functions $f(z)$ which satisfy $\mathrm{Re} f(z)/g(z) > 0$ for $|z| < 1$, when $g(z)$ is either starlike of order α or convex of order α.

In this paper we determine the distortion Theorems for functions $f(z)$ which satisfy $\mathrm{Re}(f(z)/g(z)) > \alpha$ or $\mathrm{Re}(zf'(z)/(z)) > \alpha$ for $|z| < 1$, $0 \leq \alpha < 1$. Also we find the radius of convexity for functions $f(z)$ which satisfy $\mathrm{Re}(zf'(z)/g(z)) > \alpha$ for $|z| < 1$. Further we find the radius of starlikeness for functions $f(z)$ which satisfy $\mathrm{Re}(f(z)/g(z)) > \alpha$ for $|z| < 1$, where $g(z)$ is either starlike or convex.

2. WE NEED THE FOLLOWING LEMMA

LEMMA. *Let* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ *be analytic for* $|z| < 1$ *and* $g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n$ *be analyitc and univalent on* $\{z : |z| < 1\}$.

Then

(i) $\sum_{n=2}^{\infty} |a_n| - \alpha \sum_{n=2}^{\infty} |b_n| \leq 1 - \alpha$ *provided* $\mathrm{Re}(f(z)/g(z)) > \alpha$ *for* $|z| < 1$.

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(ii) $\sum_{n=2}^{\infty} n\,|a_n| - \alpha \sum_{n=2}^{\infty} |b_n| \le 1 - \alpha$ *provided* $\mathrm{Re}\,(zf'(z)/g(z)) > \alpha$ *for* $|z| < 1$.

Part (i) *of the lemma is proved in* [3] *and similarly* (ii) *follows.*

THEOREM 1. *Let* $f(z) = z - \sum_{n=2}^{\infty} |a_n|\, z^n$ *be analytic for* $|z| < 1$ *and* $g(z) = z - \sum_{n=2}^{\infty} |b_n|\, z^n$ *be analytic and univalent on* $\{z : |z| < 1\}$. *If* $\mathrm{Re}\,(f(z)/g(z)) > \alpha$ *for* $|z| < 1$ *then*

$$r - \frac{2-\alpha}{2}\, r^2 \le |f| \le r + \frac{2-\alpha}{2}\, r^2 \qquad \text{for} \quad |z| = r.$$

Equality holds for $f(z) = z - \dfrac{2-\alpha}{2}\, z^2$ *with* $g(z) = z - \dfrac{z^2}{2}$ *for* $z = \pm r$.

Proof. From the lemma we obtain

$$\sum_{n=2}^{\infty} |a_n| \le 1 - \alpha + \alpha \sum_{n=2}^{\infty} |b_n|\,. \tag{1}$$

Since $g(z)$ is univalent on $\{z : |z| < 1\}$, $\sum_{n=2}^{\infty} n\,|b_n| \le 1$ [4]. Hence $\sum_{n=2}^{\infty} |b_n| \le \frac{1}{2}$. (1) yields

$$\sum_{n=2}^{\infty} |a_n| \le \frac{2-\alpha}{2}\,. \tag{2}$$

Also

$$|f(z)| \le r + \sum_{n=2}^{\infty} |a_n|\, r^n$$

$$\le r + r^2 \sum_{n=2}^{\infty} |a_n|$$

$$\le r + \frac{2-\alpha}{2}\, r^2.$$

Similarly

$$|f(z)| \ge r - \sum_{n=2}^{\infty} |a_n|\, r^n$$

$$\ge r - r^2 \sum_{n=2}^{\infty} |a_n|$$

$$\ge r - \frac{2-\alpha}{2}\, r^2.$$

THEOREM 2. *Let* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ *be analytic for* $|z| < 1$ *and* $g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n$ *be analytic and univalent on* $\{z : |z| < 1\}$.

If $\mathrm{Re}(zf'(z)/g(z)) > \alpha$ *for* $|z| < 1$ *then*

$$r - \frac{2-\alpha}{4} r^2 \leq |f| \leq r + \frac{2-\alpha}{4} r^2 \qquad \text{for} \quad |z| = r.$$

Equality holds for $f(z) = z - \frac{2-\alpha}{4} z^2$ *with* $g(z) = z - \frac{z^2}{2}$ *for* $z = \pm r$.

Proof. From the lemma we obtain.

$$\sum_{n=2}^{\infty} n |a_n| \leq 1 - \alpha + \alpha \sum_{n=2}^{\infty} |b_n| . \tag{3}$$

As seen before $\sum_{n=2}^{\infty} |b_n| \leq \frac{1}{2}$, (3) yields

$$\sum_{n=2}^{\infty} n |a_n| \leq \frac{2-\alpha}{2} . \tag{4}$$

Since $2 \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} n |a_n|$, from the above inequality we obtain $\sum_{n=2}^{\infty} |a_n| \leq \frac{2-\alpha}{4}$. The remaining part of the proof is similar to that of Theorem 1.

THEOREM 3. *Let* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ *be analytic for* $|z| < 1$ *and* $g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n$ *be analytic and satisfy* $\mathrm{Re}(zg'(z)/g(z)) > 0$ *for* $|z| < 1$. *Also let* $\mathrm{Re}(zf'(z)/g(z)) > \alpha$ *for* $|z| < 1$. *Then* $f(z)$ *is convex for* $|z| < r = r(\alpha) = \frac{1}{2-\alpha}$.

Proof. Since $g(z)$ is univalent, $\sum_{n=2}^{\infty} n |b_n| \leq 1$ and it follows that

$$\sum_{n=2}^{\infty} |b_n| \leq \frac{1}{n} \qquad \text{for some } n. \tag{5}$$

Using the lemma, we get from (5),

$$\sum_{n=2}^{\infty} n |a_n| \leq 1 - \frac{n-1}{n} \alpha . \tag{6}$$

Hence $\mathrm{Re}\, f'(z) > \frac{n-1}{n}\alpha$ and $f(z)$ is convex for

$$|z| < r = r(\alpha) = \inf_n \left(\frac{1}{n\left(1 - \frac{n-1}{n}\alpha\right)} \right)^{1/n-1} \quad \text{for} \quad n = 2, 3, 4, \cdots \quad [3]$$

$$= \frac{1}{2-\alpha}.$$

We have $r(0) = \frac{1}{2}$ and $r\left(\frac{1}{2}\right) = \frac{2}{3}$. The estimate is sharp for the function $f(z) = z - \frac{2-\alpha}{4} z^2$ with $g(z) = z - \frac{z^2}{2}$.

Theorem 3 is comparable to the following sharp result of Padmanabhan [2]. *Let* $f(z) = z + a_2 z^2 + \cdots$ *be analytic for* $|z| < 1$ *and* $g(z) = z + b^2 + z^2 \cdots$ *be analytic and satisfy* $\mathrm{Re}\,(zg'(z)/g(z)) < 0$ *for* $|z| < 1$. *Also let* $\mathrm{Re}\,(zf'(z)/g(z)) > \frac{1}{2}$ *for* $|z| < 1$. *Then* $f(z)$ *is convex for* $|z| < \frac{1}{2}\left[(2\sqrt{2} - 1)^{\frac{1}{2}} - (\sqrt{2} - 1)\right]$.

Remark. The functions $f(z)$ satisfying the hypothesis of Theorem 3 are close-to-convex of order α and type zero, according to the defination Libera [1].

THEOREM 4. *Let* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ *be analytic for* $|z| < 1$ *and* $g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n$ *be analytic and univalent on* $\{z : |z| < 1\}$. *If* $\mathrm{Re}\,(f(z)/g(z)) > \alpha$ *for* $|z| < 1$ *then* $f(z)$ *is univalent and starlike for* $|z| < r = r(\alpha) = \frac{1}{2-\alpha}$.

Proof. Using the lemma, we get from (5)

$$\sum_{n=2}^{\infty} |a_n| \leq 1 - \frac{n-1}{n}\alpha. \tag{7}$$

Hence $\mathrm{Re}\,(f(z)/z) > \frac{n-1}{n}\alpha$ for $|z| < 1$ and $f(z)$ is starlike in

$$|z| < r = r(\alpha) = \inf_n \left(\frac{1}{n\left(1 - \frac{n-1}{n}\right)\alpha} \right)^{1/n-1} \quad \text{for} \quad n = 2, 3, 4, \cdots \quad [3]$$

$$= \frac{1}{2-\alpha}.$$

The estimate is sharp for the function

$$f(z) = z - \frac{2-\alpha}{2} z^2 \qquad \text{with} \quad g(z) = z - \frac{z^2}{2} .$$

THEOREM 5. *Let* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ *be analytic for* $|z| < 1$ *and* $g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n$ *be analytic, univalent and convex for* $|z| < 1$. *If* $\operatorname{Re}(f(z)/g(z)) > \alpha$ *for* $|z| < 1$ *then* $f(z)$ *is univalent and starlike for* $|z| < r = (\alpha) = \inf_n \left(\frac{n}{n^2 - (n^2-1)\alpha} \right)^{1/n-1}$ *for* $n = 2, 3, 4, \cdots$.

Proof. Since $g(z)$ is convex for $|z| < 1$, $\sum_{n=2}^{\infty} n^2 |b_n| < 1$ [4]. It follows that

$$\sum_{n=2}^{\infty} |b_n| < \frac{1}{n^2} \qquad \text{for some } n. \tag{8}$$

Using the lemma, we get from (8)

$$\sum_{n=2}^{\infty} |a_n| \leq 1 - \frac{n^2-1}{n^2} \alpha . \tag{9}$$

Hence $\operatorname{Re}(f(z)/z) > \frac{n^2-1}{n^2} \alpha$ and the result follows. We have $r(0) = \frac{1}{2}$ and $r\left(\frac{1}{2}\right) = \sqrt{\frac{3}{5}}$.

The estimate is sharp for the function $f(z) = z - \left(1 - \frac{n^2-1}{n^2} \alpha\right) z^n$ with $g(z) = z - \frac{1}{n^2} z^n$ for some n.

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