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A non linear functional renewal equation

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Analisi matematica. — *A non linear functional renewal equation.* Nota [(*)] di EUGENIO SINESTRARI [(**)], presentata dal Socio G. SANSONE.

RIASSUNTO. — Si studia un'equazione funzionale non lineare proveniente dalla dinamica di popolazione. Si dimostra l'esistenza globale e l'unicità di una soluzione positiva. Inoltre si studia la dipendenza della popolazione totale dai dati.

1. INTRODUCTION

In the classical age-dependent population dynamics, initiated by Lotka and Sharpe in 1911, the population growth is studied through the age-density function $u(a,t)$ defined such that $\int_{a_1}^{a_2} u(a,t)\,\mathrm{d}a$ is the number of individuals in the age interval $[a_1, a_2]$ at time t. The mortality m and the fertility b are supposed to depend only on age a; in this case the population dynamics can be predicted if we know the birth rate $\mathrm{B}(t) = u(0,t)$: this function is the solution of a linear Volterra integral equation of convolution type (the so called renewal equation) about which there exists a large literature (see [1], [2], [3]).

In our model we take into account the dependence of mortality and fertility also on the age-density to stress the interaction of an individual with others of the same age and to consider some birth control. Variations of m and b with time t will be treated in a subsequent work.

We are thus lead to find $u(a,t) \geq 0$, solution of the following non linear version of the Lotka-Von Foerster equations:

$$(\mathrm{P}) \begin{cases} \mathrm{D}u(a,t) = -m(a, u(a,t))\,u(a,t) & 0 \leq a < \omega,\ t \geq 0 \\ u(0,t) = \int_0^\omega b(a, u(a,t))\,u(a,t)\,\mathrm{d}a & t > 0 \\ u(a,0) = p(a) & 0 \leq a < \omega \end{cases}$$

where $\mathrm{D}u(a,t) = \lim_{h\to 0} \frac{u(a+h, t+h) - u(a,t)}{h}$, p is a given initial distribution of population and ω is a least upper bound for the age of an individual. We will set $\mathrm{A} = [0, \omega[$.

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Griffel [4] studies the case in which $m(a, u) = d(a) u$, $b(a, u) = b(a)$ and $\omega = +\infty$. Under mild conditions, which are justified from a biological point of view, we prove that problem (P) can be reduced essentially to a non linear functional equation: in this paper we will prove the existence and uniqueness of the solution for each $t > 0$ and we shall study the dependence of the total population at the instant t i.e. $P(t) = \int_0^\omega u(a, t)\,\mathrm{d}a$ on the initial age distribution p; we can also prove the continuity of $P(t)$. In a subsequent paper we shall investigate the existence of stationary solutions (i.e. equilibrium age distributions) and their stability.

2. BASIC EQUATIONS

We shall assume throughout this paper that the following assumptions are satisfied for the mortality function m:

(m_1) $m(a, u)$ is continuous and non negative on $A \times R_+$; set $m_* = \inf m$.

(m_2) $u \to m(a, u)$ is non decreasing

(m_3) given $a \in A$ and $\delta > 0$, $\lim_{k \to \omega} \int_a^k m(x, u)\,\mathrm{d}u = +\infty$, uniformly for $u \leq \delta$.

Condition (m_2) has an obvious physical interpretation; (m_3) assures that $u(a, t)$ vanishes when a approaches the maximum age ω.

The fertility function b must satisfy:

(b_1) $b(a, u)$ is continuous, non negative on $A \times R_+$ and $b^* = \sup b < +\infty$.

(b_2) Given $\delta > 0$, there exists $L = L(\delta)$ such that $|b(a, u') - b(a, u'')| \leq L |u' - u''|$ for $a \in A$ and $0 \leq u', u'' \leq \delta$.

Concerning the initial age distribution $p(a)$ we shall assume that:

(p_1) p is piecewise continuous and non negative on A

(p_2) $p^* = \sup p$ and $P_0 = \int_0^\omega p(a)\,\mathrm{d}a$ are finite.

The last condition guarantees that the total population remains finite. We shall also require that the solution $u(a, t)$ of (P) is non negative and such that $u(0, t)$ is continuous for $t > 0$ and $u(0, 0^+)$ is finite.

Given an interval I such that $\min I = 0$ and a continuous function $B : I \to R_+$, let us consider the problem:

$$
(\mathrm{P}') \quad \begin{cases} \mathrm{D}u(a, t) = -m(a, u(a, t))\, u(a, t) & a \in A, t \in I \\ u(0, t) = B(t) & t \in I, t > 0 \\ u(a, 0) = p(a) & a \in A. \end{cases}
$$

We can easily obtain the following

PROPOSITION 1.

Let u_B *be a solution of* (P′) *on* A×I. *If* B *verifies for* $t \in I, t > 0$

$$B(t) = \int_0^{\omega} b(a, u_B(a, t))\, u_B(a, t)\, da \tag{1}$$

then $u = u_B$ *is a solution to* (P). *Conversely if* u *is a solution of* (P) *on* A×I, *setting* $B(t) = u(0, t), t \in I, t > 0$ *we obtain that* B *is continuous on* I, *verifies* (1) *with* $u_B = u$ *and* u *is a solution of* (P′).

The proof is obvious.

If we can get a unique solution u_B to (P′) for a general continuous functions B then the right hand side of (1) is a non linear functional of B (as the dependence of u_B from B cannot be written in closed form for m verifying only conditions $(m_1) - (m_3)$) and because of proposition 1 problem (P) is reduced to find a function B which satisfies the non linear functional renewal equation (1).

Problem (P′) can be easily solved for a general B by the method of characteristics and owing to (m_1) we have a global solution.

THEOREM 1. *Let* $B : R_+ \to R_+$ *be a given continuous function. There is a unique* $u_B(a, t) \geq 0$ *which satisfies* (P′) *on* $A \times R_+$. *Moreover we have*:

$$u_B(a, t) = \begin{cases} B(t - a) \exp\left(-\int_0^a m(x, u_B(x, t - a + x))\, dx\right) & a < t, a \in A \\ p(a - t) \exp\left(-\int_0^t m(a - t + x, u_B(a - t + x, x))\, dx\right) & 0 \leq t \leq a < \omega. \end{cases}$$

and $\lim_{h \to \omega - a} u_B(a + h, t + h) = 0$ for $(a, t) \in A \times R_+$.

The last result is obtained by using condition (m_3).

3. A PRIORI ESTIMATES

In this section we shall derive some a-priori estimates for a solution u of (P), for the birth rate $u(0, t)$ and for the total population $P(t)$; these estimates have biological implications and let us find the suitable class of functions $B(t)$ where it is possible to study equation (1) as will be shown in the next section.

THEOREM 2. *Set* $\mathrm{d} = b^* - m_*$, $\mathrm{P}_1 = b^* \mathrm{P}_0$ *and* $c = \max(\mathrm{P}_1, p^*)$. *If* u *is a solution to* (P) *then the following estimates hold*:

$$(2) \qquad \begin{cases} u(a, t) \leq c \exp(\mathrm{d}t) \\ u(0, t) \leq \mathrm{P}_1 \exp(\mathrm{d}t) \\ \mathrm{P}(t) \leq \mathrm{P}_0 \exp(\mathrm{d}t) . \end{cases}$$

Proof. Set $u(0, t) = \mathrm{B}(t)$. If $t < \omega$ by using Theorem 1 we obtain $\mathrm{B}(t) = \int_0^t b(a, u) u \, \mathrm{d}a + \int_t^\omega b(a, u) u \, \mathrm{d}a \leq b^* \left(\int_0^t u(a, t) \, \mathrm{d}a + \int_t^\omega u(a, t) \, \mathrm{d}a \right) \leq$ $\leq b^* \left(\int_0^t \mathrm{B}(t - a) \exp(-m_* a) \, \mathrm{d}a + \int_t^\omega p(a - t) \exp(-m_* t) \, \mathrm{d}a \right)$; from this follows $\mathrm{B}(t) \exp(m_* t) \leq b^* \int_0^t \mathrm{B}(a) \exp(m_* a) \, \mathrm{d}a + b^* \int_0^\omega p(a) \, \mathrm{d}a$; this inequality is verified also if $t \geq \omega$ and $t \in \mathrm{I}$; then from Gronwall's inequality we get (2_2). (2_1) and (2_3) are easy consequences of Theorem 1 and (2_2).

As a consequence of (2_1) we have that a solution of (P) is bounded on bounded sets. Moreover the coefficient d in (2) is determined (as in the logistic model) by the relation between the mortality and the fertility.

Now let u and $\bar{u}$ be solutions of (P) with initial conditions p and $\bar{p}$ respectively. Set $u(0, t) = \mathrm{B}(t)$ and $\bar{u}(0, t) = \overline{\mathrm{B}}(t)$; from (m_2) it is possible to get:

$$(3) \qquad \int_0^\omega |u(a, t) - \bar{u}(a, t)| \, \mathrm{d}a \leq \int_0^t |\mathrm{B}(s) - \overline{\mathrm{B}}(s)| \, \mathrm{d}a + \int_0^\omega |p(a) - \bar{p}(a)| \, \mathrm{d}a.$$

From this inequality it is possible to obtain the continuity of the total population $\mathrm{P}(t)$.

4. GLOBAL EXISTENCE OF THE SOLUTION

To solve problem (P) we shall find a solution of (1) in a suitable class of admissible functions, suggested by the a priori estimates of section 3. Let us consider for each $\mathrm{T} > 0$ the set $\mathscr{B}_\mathrm{T} = \{\mathrm{B}(t)$ is continuous on $[0; \mathrm{T}]$ and $0 \leq \mathrm{B}(t) \leq \mathrm{P}_1 \exp(\mathrm{d}t), 0 \leq t \leq \mathrm{T}\}$; given $\mathrm{B} \in \mathscr{B}_\mathrm{T}$ let $u_\mathrm{B}(a, t)$ be the solusolution of (P′) and let $\mathrm{B} \to \mathrm{S}(\mathrm{B})$ be defined by:

$$\mathrm{S}(\mathrm{B})(t) = \int_0^\omega b(a, u_\mathrm{B}(a, t)) \, u_\mathrm{B}(a, t) \, \mathrm{d}a \qquad 0 \leq t \leq \mathrm{T}.$$

THEOREM 3. *For sufficiently small* T, S(B) *is a contraction mapping on* $\mathcal{B}_T$ *and problem* (P) *has a unique solution on* $A \times [0, T]$.

Proof. With the aid of (2) and (b_2) we can prove that $S(B) \in \mathcal{B}_T$. From (b_2) there is $L > 0$ such that $|b(a, u_1) - b(a, u_2)| \leq L |u_1 - u_2|$ when $a \in A$ and $0 \leq u_1, u_2 \leq c \exp(|d|)$. Take $T \in]0, 1[$ such that $T(b^* + Lc \exp(|d|)) < 1$. If $B_1, B_2 \in \mathcal{B}_T$ we get that $u_{B_1}(a, t), u_{B_2}(a, t) \leq \leq c \exp(|d|)$ and $u_{B_1} \equiv u_{B_2}$ when $a \geq t$. Hence for $t \in [0, T]$ we have:

$$|S(B_1)(t) - S(B_2)(t)| \leq \int_0^t b(a, u_{B_1}) |u_{B_1} - u_{B_2}| \, da +$$

$$+ \int_0^t |b(a, u_{B_1}) - b(a, u_{B_2})| \, u_{B_2} \, da \leq b^* \int_0^t |u_{B_1} - u_{B_2}| \, da +$$

$$+ L \int_0^t |u_{B_1} - u_{B_2}| \, u_{B_2} \, da \leq (b^* T + Lc \exp(|d|) T) \sup_{0 \leq a \leq t \leq T} |u_{B_1} - u_{B_2}| \leq$$

$$\leq T(b^* + Lc \exp(|d|) \sup_{0 \leq t \leq T} |B_1(t) - B_2(t)| .$$

and the conclusion follows.

By using the a priori estimates it is also possible to extend this solution to all $A \times R_+$:

THEOREM 4. *There exists a unique solution to* (P) *for each* $a \in A$ *and* $t \in R_+$.

Proof. Let u be the maximally defined solution of (P) in $A \times [0, T[$. If $T < +\infty$ we get a contradiction as u could be extended as a solution in $A \times [0, T]$. This can be proved through the results of section 3.

From (b_2) it is possible to derive the following result about the dependence of the total population and of the birth rate on the initial age distribution p.

THEOREM 5. *Let* $\delta^* > 0$ *and* $\sup I = T^* < +\infty$. *Let us choose* p *and* $\bar{p}$ *verifying* $(p_1), (p_2)$ *and such that* $p^*, \bar{p}^*, P_0, \overline{P}_0 < \delta^*$. *Let* u *and* $\bar{u}$ *be solutions of* (P) *on* $A \times I$ *with initial conditions* p *and* $\bar{p}$ *respectively. Then we have for* $t \in I$

$$(4) \qquad \begin{aligned} &|P(t) - \overline{P}(t)| \leq \exp(c^* t) \, \|p - p\|_{L^1(A)} \\ &|u(0, t) - \bar{u}(0, t)| \leq c^* \exp(c^* t) \, \|p - p\|_{L^1(A)} \end{aligned}$$

where c^* *depends on* δ^* *and* T^*.

Proof. Set $u(0, t) = B(t)$ and $\bar{u}(0, t) = \overline{B}(t)$. From (b_2) we get

$$\int_0^t |B(s) - \overline{B}(s)| \, ds = \int_0^t ds \left| \int_0^\omega (b(a, u(a, s)) u(a, s) - \right.$$

$$\left. - b(a, \bar{u}(a, s)) \bar{u}(a, s)) \, da \right| \le c^* \int_0^t ds \int_0^\omega |u(a, s) - \bar{u}(a, s)| \, da .$$

It we substitute in (3) and use Gronwall's inequality we get

$$\int_0^\omega |u(a, t) - \bar{u}(a, t)| \, da \le \exp(c^* t) \, \|p - \bar{p}\|_{L^1(A)} .$$

In the same way we find the second estimate of (4).

Remark. Let us observe that in some examples of biological relevance $u \to b(a, u) u$ is Lipschitz continuous uniformly for $a \in A$ and $u \ge 0$; take for example $b(a, u) = f(a)(1 + u^\alpha)^{-1}$ with f bounded on A and $\alpha \ge 0$; in this case Theorem 5 holds with c^* independent on δ^* and T^*.

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