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On the existence of a periodic solution of n-th order nonlinear differential equations

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Equazioni differenziali ordinarie. — *On the existence of a periodic solution of n-th order nonlinear differential equations.* Nota di B. MEHRI e H. A. EMAMIRAD, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Gli Autori provano un Teorema di esistenza di una soluzione periodica di una classe di equazioni differenziali nonlineari ordinarie.

1. In a previous paper with the same title [1], the authors announced some results concerning the existence of a periodic solution a for certain class of n-th order non-autonomous nonlinear differential equations. The object of the paper is to obtain similar results in some different setting.

Consider the following nonlinear n-th order differential equation

$$x^{(n)} + f(t, x, x', \cdots, x^{(n-1)}) = e(t) \tag{1}$$

where f and e are continuous and periodic with respect to t of period ω, and

$$\int_0^\omega e(t)\, dt = 0 . \tag{2}$$

Further, we suppose that all the solutions of (1) with n initial conditions are unique (so it would suffice to take f to be C^1 in each variable).

It will be shown that under some conditions on f, there exists at least one solution of (1) satisfying the periodic boundary conditions

$$x^{(i)}(0) = x^{(i)}(\omega) \quad , \quad i = 0, 1, 2, \cdots, n-1 . \tag{3}$$

The method used here is similar to that employed by Lazer [2]. Finally we give applications in forcing Van der Pol equation and also certain typical third-order nonlinear equations. The results obtained are in fact a generalization of [3], [4] and [5].

2. We define the Green's function of the equation $x^{(n)} = \frac{1}{\omega}$ with periodic boundary condition (3) as follows:

$$G(t) = \begin{cases} \sum_{k=0}^{n-1} a_k t^k + \dfrac{t^n}{n!\,\omega} & \text{for} \quad x \leq t < s \leq \omega \\ \sum_{k=0}^{n-1} b_k t^k + \dfrac{t^n}{n!\,\omega} & \text{for} \quad 0 \leq s < t \leq \omega \end{cases}$$

(*) Nella seduta del 14 giugno 1979.

where we have the following relations between the coefficients a_k and b_k:

(4) $$a_{n-k} - b_{n-k} = \frac{(-1)^{k+1} s^{k-1}}{(k-1)!\,(n-k)!}, \qquad k = 1, 2, \cdots, n-1$$

the relations show that $G(t, s)$ is continuous as well as its derivatives up to the order $n-2$. For $k = 1$, we have

(5) $$(n-1)!\,(a_{n-1} - b_{n-1}) = 1$$

which establishes easily

(6) $$\frac{\partial^{n-1}}{\partial^{n-1}} G(t, t-) - \frac{\partial^{n-1}}{\partial t^{n-1}} G(t, t+) = -1 .$$

We also need to have

(7) $$\frac{\partial^k}{\partial t^k} G(t, s)\,|_{t=0} = \frac{\partial^k}{\partial t^k} G(t, s)\,|_{t=\omega}, \qquad k = 0, 1, \cdots, n-1$$

thus, $(b_1, b_2, \cdots, b_{n-1})$ ought to be the solution of the following linear system

$$\begin{aligned} b_1 + \omega b_2 + \cdots + \omega^{n-2} b_{n-1} &= \beta_0 \\ 2\,\omega b_2 + \cdots + (n-1)\,\omega^{n-3} b_{n-1} &= \beta_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ (n-1)!\, n_{n-1} &= \beta_{n-2} \end{aligned}$$

where

$$\beta_i = \frac{(-1)^{n-i}(n-i+1)\,s^{n-i} - \omega^{n-i}}{(n-i+1)!\,\omega}, \qquad i = 1, 2, \cdots, n-1 .$$

For $k = n-1$, the equation (7) leads us again to the relation (5). Thus, the $2n$ coefficients a_k, b_k can be determined by $2n-1$ equations and the solution always exists.

3. Main Theorem. Assume f satisfies the following conditions

i) For some non-negative real number b and for all $t \in [0, \omega]$

(8) $$x_i f(t, x_1, x_2, \cdots, x_n) \geq 0 \qquad i = 1, 2, \cdots, n$$

when all $x_i \geq b$ or all $x_i \leq -b$

ii) There exists a constant D, such that

(9) $$b + 3m \leq D$$

where

$$m = \mathrm{Max}\,\{M, M_0(M+E), \cdots, \omega M_{n-1}(M+E)\}$$

and

$$\mathrm{M} = \mathrm{Max}\,\{\,|f(t\,,x_1\,,\cdots,x_n)\,|\,;t\in[0\,,\omega]\,,\,|\,x_i\,|\leq \mathrm{D}\}\,,\qquad i=1\,,2\,,\cdots,n$$

$$\mathrm{E} = \mathrm{Max}\,\{\,|\,e(t)\,|\,,t\in[0\,,\omega]\}$$

$$\mathrm{M}_i = \mathrm{Max}\left\{\left|\frac{\partial^i}{\partial t^i}\,\mathrm{G}(t\,,s)\,,(t\,,s)\in[0\,,\omega]\times[0\,,\omega]\right.\right\}.$$

Then equation (1) has at least one solution $x(t)$ satisfying the periodic boundary condition (3).

Proof. Let us consider the following integral equation

$$x(t)=\int_0^\omega \mathrm{G}(t\,,s)\,\{f(s\,,x(s)\,,\cdots,x^{(n-1)}(s))-e(s)\}\,\mathrm{d}s\,. \tag{10}$$

Obviously, the solution of the above integral equation satisfies not only the periodic boundary conditions (3) but also

$$x^{(n)}+f(t\,,x\,,\cdots,x^{(n-1)})=e(t)+\frac{1}{\omega}\int_0^\omega f(s\,,x(s)\,,\cdots,x^{(n-1)}(s))\,\mathrm{d}s\,.$$

In what follows, we shall prove our principal integral equation (10) has a solution $x(t)$ satisfying

$$\int_0^\omega f(s\,,x(s)\,,x'(s)\,,\cdots,x^{(n-1)}(s))\,\mathrm{d}s=0\,. \tag{11}$$

Let $\mathrm{C}^k[0\,,\omega]$ be the space of all k-th differentiable functions on $[0\,,\omega]$ equipped with the following norm

$$\|x\|_k=\sum_{i=0}^{k}\sup_{t\in[0,\omega]}|\,x^{(i)}(t)\,|$$

and let $\mathrm{B}=\left(\prod_{k=0}^{n-1}\mathrm{C}^k[0\,,\omega]\right)\times\mathbf{R}$, with the norm

$$|\,(x\,,x',\cdots,x^{(n-1)}\,,a)\,|=\|x\|_{n-1}+|\,a\,|\,.$$

With this definition B is a complete normed linear space.

Now let us define the following operator on B.

$$\mathrm{T}(x\,,x',\cdots,x^{(n-1)}\,,a)=(x^*,x'^*\,,\cdots,x^{(n-1)*},a^*)$$

where, for each $i=0\,,1\,,2\,,\cdots,n-1$

$$x^{(i)*}(t)=a+\int_0^\omega\frac{\partial^i}{\partial t^i}\,\mathrm{G}(t\,,s)\,\{f(s\,,x(s)\,,\cdots,x^{(n-1)}(s)-e(s)\}\,\mathrm{d}s \tag{12}$$

and

$$a^* = a - \frac{1}{\omega} \int_0^\omega f(s, x^*(s), \cdots, x^{(n-1)*}(s))\, ds . \tag{13}$$

The operator T is a continuous mapping from B into B. Now, let us define a closed convex and bounded subset of B as follows

$$K = \{(x, x', \cdots, x^{(n-1)}, a) \in B ;$$
$$\| x \|_k \leq D \;,\; k = 0, 1, 2, \cdots, n-1 \;,\; | a | \leq b + 2m\}$$

Now, if we show T has a fixed point in K, then from (13) we can deduce (11), which ends the proof of our main theorem. In order to prove this fact we apply Schauder's fixed point theorem. In fact we have, for $(x, x', \cdots, x^{(n-1)}, a) \in K$

$$| x^{(i)*}(t) | \leq | a | + M_i (E + M) \leq b + 2m + m \leq D .$$

On the other hand if $| a | \leq b + m$

$$\left| \frac{1}{\omega} \int_0^\omega f(s, x^*(s), x'^*(s), \cdots, x^{(n-1)*}(s)\, ds \right| \leq M \leq m$$

which leads to $a\, a^* | \leq b + 2m$.

Now, we consider the two other cases, first $b + m < a \leq b + 2m$, in which (12) implies

$$\sup_{t \in [0, \omega]} | x^{(i)*}(t) - a | \leq M_i (E + M) \leq m$$

the above estimate together with $b + m$ a leads to $\sup x^{(i)*}(t) \geq b$, $t \in [0, \omega]$

$$f(t, x^*(t), x'^*(t), \cdots, x^{(n-1)*}(t)) \geq 0$$

for all $t \in [0, \omega]$. Taking (13) into account we obtain $b \leq a^* \leq b + 2m$.

And second $-(b + 2m) \leq a \leq -(b + m)$ with a similar argument we will have

$$f(t, x^*(t), x'^*(t), \cdots, x^{(n-1)*}(t)) \leq 0$$

for all $t \in [0, \omega]$, which leads to $-(b + 2m) \leq a^* \leq -b$, thus in any case $| a^* | \leq b + 2m$ and therefore $T(B) \subset B$.

Finally T is a compact operator, since for any infinite sequence in T (B) one can extract a convergent subsequence, by remarking that the norm of the functions

$$V_i(t) = \int_0^\omega \frac{\partial^i}{\partial t^i} G(t, s) \left[f(s, x(s), \cdots, x^{(n-1)}(s)) - e(s) \right] ds$$

are bounded for $i = 0, 1, 2, \cdots, n$ and applying the Ascoli's lemma.

4. Applications

In this section, we shall consider some applications of the main theorem

(A 1) First consider the forced Van der Pol's equation

$$x'' + \varepsilon x' (x^2 - 1) + ax = \mu \cos \pi t \tag{14}$$

where ε, a and μ are convienent real positive constants. In order to prove the existence of a 2–periodic solution for (14), we define the correspondent Green's function

$$G(t, s) = \frac{1}{2\omega} \begin{cases} \left(s - t - \frac{\omega}{2}\right)^2 & 0 \leq t < s \leq \omega \\ \left(t - s - \frac{\omega}{2}\right)^2 & 0 \leq s < t \leq \omega \end{cases}$$

choose $b = 1$ in the main theorem and

$$\begin{aligned} M &= \operatorname{Max} \{ | \varepsilon x' (x^2 - 1) + ax | \; ; \; |x| \leq D , |x'| \leq D \} \\ &\leq \varepsilon (D^3 - D) + aD \end{aligned}$$

obviously $M_0 = \frac{\omega}{8}$, $M_1 = 1$ and therefore $m = \varepsilon D^3 - \varepsilon D + aD + \mu$, thus for establishing (9) we ought to have

$$1 + 3 (\varepsilon D^3 - \varepsilon D + aD +) \leq D$$

which is valid if one takes ε, a and μ sufficiently small and $D > 1$.

(A 2) In the paper [6] Reissig proved the existence of at least one periodic solution of period ω for the equation

$$x''' + \phi (x') x'' + K^2 x' + f(x) = \mu p (t) \tag{15}$$

and in [1] authors have obtained the similar result when $\| p \| = 1$.

In this regard we can establish the following theorem which is the direct consequence of the main theorem.

THEOREM 1. *Equation* (15) *admits at least* 1*-periodic solution if*

a) $p(t + 1) = p(t)$ and $\int_0^1 p(t) \, dt = 0$

b) $\phi(x')$ is a non-negative bounded C^1 function

c) f is a C^1 and $x f(x) > 0$, for all x

d) for some $D > 0$

$$(L + K^2) + \frac{f(D)}{D} + \frac{|\mu|}{D} \leq \frac{4}{19} \tag{16}$$

where L *is an upper bound of* ϕ.

Proof. Here we define the Green's function as:

$$G(t,s)=\begin{cases}\dfrac{t^3}{6}-\dfrac{t^2}{2}\left(s-\dfrac{1}{2}\right)+t\left(\dfrac{s^2}{2}-\dfrac{s}{2}+\dfrac{1}{12}\right)+\dfrac{s^2}{2} & \text{for } 0\le t<s\le 1\\[2ex] \dfrac{t^3}{6}-\dfrac{t^2}{2}\left(s+\dfrac{1}{2}\right)+t\left(\dfrac{s^2}{2}+\dfrac{s}{2}+\dfrac{1}{12}\right) & \text{for } 0\le s<t\le 1\end{cases}$$

obviously $M_0=\frac{3}{2}$, $M_1=\frac{19}{12}$, $M_2=\frac{1}{2}$ and $M=(L+K^2)D+f(D)$, therefore by taking $b=0$, $m=\frac{19}{12}((L+K^2)D+f(D)+|\mu|)$. Thus, condition

(i) of the main theorem will be satisfied if (16) holds and by virtue of c) the condition (i) is also guaranteed

(A 3) An existence theorem for the following equation is studied by Reissig [7] and Ezeilo [8]

$$x'''+\phi(x')x''+\psi(x)x'+\theta(t,x,x',x'')=\mu p(t)\,. \tag{17}$$

The following theorem insures the existence of 1-periodic solution for (17), assuming somme different conditions

THEOREM 2. *Assumption a) and b) of the previous theorem hold on the functions p and ϕ, furthermore*

c') ψ *is a non-negative* K-*bounded* C^1 *function*

d') *For some* $D>0$

$$(L+K)+\frac{F}{D}+\frac{|\mu|}{D}\le\frac{4}{19}$$

e) $\theta(t,x,x',x'')$ *is* 1-*periodic function in* t *and* C^1 *in* x, x', x'' $\theta(t,x,x',x'')\operatorname{sign}x^{(i)}\ge 0$ for all $i=0,1,2$ *and finally* $F=\operatorname{Max}\{|\theta(t,x,x',x'')|:|x^{(i)}|\le D,\ i=0,1,2\}$.

The proof of the previous theorem can be adapted for this one.

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