Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Olusola Akinyele

Necessary Conditions for the non-uniform partial stability for delay systems

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **66** (1979), n.6, p. 509–515.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1979_8_66_6_509_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1979.

Equazioni differenziali ordinarie. — *Necessary Conditions for the non-uniform partial stability for delay systems.* Nota di OLUSOLA AKINYELE, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore dà condizioni necessarie per la stabilità parziale non uniforme di un sistema di equazioni differenziali con ritardo.

§ 1. INTRODUCTION

In a very recent paper, Corduneanu [2] investigated some problems of partial stability of linear differential equations with delay. He gave a characterization of the partial asymptotic exponential stability of the trivial solution of the linear system and then obtained conditions of partial stability of the trivial solution for certain perturbed systems.

In the present paper, we wish to extend the characterization in [2] to non-uniform partial asymptotic exponential stability of a non-linear differential system with delay relative to an invariant set. Precisely, we introduce the notion of asymptotically-self invariant set with respect to some part of the variables and then carry out our investigations relative to such invariant sets. Finally, we also give necessary conditions for the concept of generalized partial asymptotic exponential stability of the invariant set in terms of Lyapunov functional. The results of this paper estend and generalize some results of [2]. In conclusion the investigations of this paper can be improved by searching simultaneously for informations on the behaviour of both components of the solution of the delay system.

§ 2. PRELIMINARIES AND MAIN RESULTS

We shall consider the functional differential system

$$(1) \qquad \begin{cases} \dot{v}(t) = f(t, v_t, w_t) \\ \dot{w}(t) = g(t, v_t, w_t) \end{cases} \qquad v_{t_0} = \phi_0, \qquad w_{t_0} = \psi_0$$

where $t \in \mathrm{R}^+ = [0, \infty)$, $v \in \mathrm{R}^n$, $w \in \mathrm{R}^m$ and f, g are continuous functions from $\mathrm{R}^+ \times \mathrm{C}([-h, 0], \mathrm{R}^n) \times \mathrm{C}([-h, 0], \mathrm{R}^m)$ into R^n and R^m respectively, $\mathrm{C}([-h, 0], \mathrm{R}^n)$ being the space of continuous functions from $[-h, 0]$ into

(*) Nella seduta del 21 aprile 1979.

R^n. Here for $h > 0$, $l^n = C([-h, 0], R^n)$ and $\|\phi\|_0 = \max_{-h \leq s \leq 0} \|\phi(s)\|$, $\|\cdot\|$ being convenient norm in R^n. If $(t_0, \phi, \psi) \in R^+ \times l^n \times l^m$, we denote by $v_t(t_0; \phi, \psi)$ and $w_t(t_0; \phi, \psi)$ the solution of (1) such that $v_{t_0} = \phi$ and $w_{t_0} = \psi$. For $t \geq t_0$, $v_t(t_0; \phi, \psi) \in l^n$ and $w_t(t_0; \phi, \psi) \in l^m$.

DEFINITION 2.1. A function $\alpha \in C(R^+, R^+)$ is said to belong to class $\mathscr{L}$ if $\alpha(t)$ is decreasing in t and $\lim_{t \to \infty} \alpha(t) = 0$.

DEFINITION 2.2. The set $\phi = 0$, $\psi = 0$ is said to be

(i) ϕ-asymptotically self invariant (ϕ-ASI) with respect to the system (1) if there exists a $q \in \mathscr{L}$ such that

$$\|v_t(t_0; 0, 0)\|_0 \leq q(t_0), \qquad t \geq t_0;$$

(ii) ψ-asymptotically self-invariant (ψ-ASI) with respect to the system (1) if there exists a $q_1 \in \mathscr{L}$ such that

$$\|w_t(t_0; 0, 0)\|_0 \leq q_1(t_0), \qquad t \geq t_0$$

and

(iii) asymptotically self-invariant with respect to the system (1) if there exists $\alpha \in \mathscr{L}$ such that

$$\|v_t(t_0; 0, 0)\|_0 + \|w_t(t_0; 0, 0)\|_0 \leq \alpha(t_0), \qquad t \geq t_0.$$

Clearly (i) and (ii) imply (iii) and viceversa.

Throughout this paper, we shall assume that the set $\phi = 0$, $\psi = 0$ is ϕ-ASI with respect to the system (1).

DEFINITION 2.3. The ϕ-ASI set $\phi = 0$, $\psi = 0$ of the system (1) is said to be partially equistable or equistable with respect to the v-component if for each $t \in R^+$

$$\|v_t(t_0; \phi_0, \psi_0)\|_0 \leq K(t_0, \tau)(\|\phi_0\|_0 + \|\psi_0\|_0) + q(t_0); \qquad t \geq t_0$$

where $K \in C(R^+ \times R^+, R^+)$ and $q \in \mathscr{L}$.

DEFINITION 2.4 The ϕ-ASI set $\phi = 0$, $\psi = 0$ of the system (1) is said to be partially equi-exponential asymptotically stable or equi-exponential asymptotically stable with respect to the v-component if for each $t_0 \in R^+$, there exist $K(t_0, \tau) > 0$ $H(t, t_0)$ and $\alpha > 0$ such that

$$\|v_t(t_0; \phi, \psi)\|_0 \leq K(t_0, \tau)(\|\phi_0\|_0 + \|\psi_0\|_0)\, e^{-\alpha(t-t_0)} + H(t, t_0) \qquad t \geq t_0,$$

where $K \in C(R^+ \times R^+, R^+)$, $H \in C(R^+ \times R^+, R^+)$ with $H(t, t) = 0$, $H(t, t_0) \leq p(t_0)$ for $p \in \mathscr{L}$ and $\lim_{t \to \infty} [\operatorname{Sup}_{t_0 \geq T} H(t, t_0)] = 0$ for some positive number T.

DEFINITION 2.5. The ϕ-ASI set $\phi = 0$, $\psi = 0$ of the system (1) is said to be generalized partially equi-exponential asymptotically stable or generalized equi-exponential asymptotically stable with respect to the v-component if for each $t_0 \in R^+$, there exists $K(t_0, \tau) > 0$ and another function $p \in \mathcal{K} = \{$class of functions $\sigma \in C([0, \rho), R^+)$ such that $\sigma(0) = 0$ and $\sigma(t)$ is strictly monotone increasing in $t\}$ for $t \in R^+$ with $p(t) \to \infty$ as $t \to \infty$ such that

$$\| v_t(t_0 ; \phi ; \psi) \|_0 \leq K(t_0, \tau)(\| \phi_0 \|_0 + \| \psi_0 \|_0) e^{(p(t_0)-p(t))} + H(t, t_0) \quad \text{for} \quad t \geq t_0 .$$

where K and H are defined as in Definition 2.4.

In particular, if $p(t) = \alpha t$, $\alpha > 0$ then definition 2.5 reduces to definition 2.4.

Remarks. It is obvious that definition 2.4 implies definition 2.3. If $q(t_0) \equiv 0$, then definition 2.3 reduces to the partial equistability definition of the trivial solution of the system (1), which follows from such definition for ordinary differential equations [3, 4]. If $H(t, t_0) \equiv 0$, $K(t_0, \tau) = K > 0$, and f, g are both linear in ϕ and ψ, then definition 2.4 reduces to the partial exponential asymptotic stability definition of the trivial solution of the system (1) [2]. If $H(t, t_0) \equiv 0$, $K(t_0, \tau) = K > 0$, then definition 2.5 reduces to the generalized partial asymptotic exponential stability definition of the trivial solution of the system [1].

DEFINITION 2.6. The ASI set $u = 0$ of the scalar differential equation

$$u' = g(t, u) \qquad u(t_0) = u_0 > 0 \tag{3}$$

where $g \in C(R^+ \times R^+, R^+)$ is said to be equistable if for each $t_0 \in R^+$, there exist $M(t_0, \tau) > 0$ and $q \in \mathcal{L}$ such that

$$u(t, t_0, u_0) \leq M(t_0, \tau) u_0 + q(t_0), \qquad t \geq t_0 .$$

We shall now give a result which is important and useful whenever we consider perturbations of systems. The result gives a characterization of the partial equi-exponential asymptotic stability of the ϕ-ASI $\phi = 0$, $\psi = 0$ in terms of Lyapunov functionals.

THEOREM 2.7. *Assume that*

(i) *the* ϕ-ASI *set* $\phi = 0$, $\psi = 0$ *of the system* (1) *is partially equi-exponentially asymptotically stable and any two solutions*

$$(v_t(t_0 ; \phi_1, \psi_1), w_t(t_0 ; \phi_1, \psi_1))$$

and

$$(v_t(t_0 ; \phi_2, \psi_2), w_t(t_0 ; \phi_2, \psi_2))$$

satisfy

$$\| v_t(t_0 ; \phi_1, \psi_1) - v_t(t_0 ; \phi_2, \psi_2) \|_0 \leq K(t_0, \tau)(\|\phi_1 - \phi_2)\|_0 + \|\psi_1 - \psi_2\|_0) e^{-\alpha(t-t_0)}$$
$$t \geq t_0 ;$$

(ii) *the function* $H(t, t_0)$ *is partially differentiable with respect to* t_0 *and*

$$\operatorname*{Sup}_{\delta \geq 0} \left\{ -\frac{\partial H}{\partial t_0}(t+\delta, t)\, e^{\alpha\delta} \right\} \leq \eta(t)$$

where $\eta \in C(R^+, R^+)$ *and* $\int_t^{t+1} \eta(s)\, ds \to 0$ *as* $t \to \infty$.

Then there exists a Lyapunov functional $W(t, \phi, \psi)$ such that

(I) $W \in C(R^+ \times C_\rho \times C_\beta, R^+)$ and W is Lipschitzian in ϕ, ψ for the function $K(t, \tau)$, where $C_\rho = \{\phi \in l^n : \|\phi\|_0 < \rho\}$ and $C_\beta = \{\psi \in l^m : \|\psi\|_0 < \beta\}$.

(II) $\|\phi\|_0 \leq W(t, \phi, \psi) \leq K(t, \tau)(\|\phi\|_0 + \|\psi\|_0)$ $(t, \phi, \psi) \in R^+ \times C_\rho \times C_\beta$; and (III)

$$D^+ W(t, \phi, \psi) = \lim_{\delta \to 0} \operatorname{Sup} \frac{1}{\delta} [W(t+\delta, v_{t+\delta}(t; \phi, \psi), w_{t+\delta}(t; \phi, \psi)) -$$

$$- W(t, \phi, \psi)] \leq -\alpha W(t, \phi, \psi) + \eta(t), (t, \phi, \psi) \in R^+ \times C_\rho \times C_\beta .$$

Proof. Define a functional $W(t, \phi, \psi)$ as follows:

$$W(t, \phi, \psi) = \operatorname*{Sup}_{\delta \geq 0} \{\|v_{t+\delta}(t; \phi, \psi)\|_0 - H(t+\delta, t)\}\, e^{\alpha\delta}$$

where $v_t(t_0; \phi, \psi), w_t(t_0; \phi, \psi)$ is any solution of the system (1).

If $\delta = 0$, $\|v_t(t_0; \phi, \psi)\|_0 \leq W(t, \phi, \psi)$, that is, $\|\phi\|_0 \leq W(t, \phi, \psi)$ and by definition $W(t, \phi, \psi) \leq K(t, \tau)(\|\phi\|_0 + \|\psi\|_0)$.

By uniqueness, $v_{t+h+\delta}(t+h; v_{t+h}(t; \phi, \psi), w_{t+h}(t; \phi, \psi)) = v_{t+h+\delta}(t; \phi, \psi)$, hence

$$D^+ W(t, \phi, \psi) =$$

$$= \lim_{h \to 0^+} \operatorname{Sup} \frac{1}{h} [\operatorname*{Sup}_{\delta \geq 0} \{ \|v_{t+h+\delta}(t+h; v_{t+h}(t; \phi, \psi), w_{t+h}(t; \phi, \psi))\|_0 -$$

$$- H(t+h+\delta, t+h)\} e^{\alpha\delta} - \operatorname*{Sup}_{\delta \geq 0} \{\|v_{t+\delta}(t; \phi, \psi)\|_0 - H(t+\delta, t)\} e^{\alpha\delta}]$$

$$= \lim_{h \to 0^+} \operatorname{Sup} \frac{1}{h} [\operatorname*{Sup}_{\delta \geq h} \{\|v_{t+\delta}(t; \phi, \psi)\|_0 - H(t+\delta, t+h)\} e^{\alpha(\delta-h)} -$$

$$- \operatorname*{Sup}_{\delta \geq 0} \{\|v_{t+\delta}(t; \phi, \psi)\|_0 - H(t+\delta, t)\} e^{\alpha(\delta-h)}$$

$$\leq \lim_{h \to 0^+} \operatorname{Sup} \frac{1}{h} [\operatorname*{Sup}_{\delta \geq 0} \{\|v_{t+\delta}(t; \phi, \psi)\|_0 - H(t+\delta, t+h)\} e^{\alpha(\delta-h)} -$$

$$- \operatorname*{Sup}_{\delta \geq 0} \{\|v_{t+\delta}(t; \phi, \psi)\|_0 - H(t+\delta, t)\} e^{\alpha\delta}] \leq$$

$$\leq W(t, \phi, \psi) \lim_{h\to 0^+} \operatorname{Sup} \frac{1}{h} (e^{-\alpha h} - 1) + \operatorname{Sup}_{\delta\geq 0} [\lim_{h\to 0^+} \operatorname{Sup} \frac{1}{h} \{H(t+\delta, t) - H(t+\delta, t+h)\}] e^{\alpha(\delta - h)}$$

$$\leq -\alpha W(t, \phi, \psi) + \operatorname{Sup}_{\delta\geq 0} \left\{ -\frac{\partial H}{\partial t_0} (t+\delta, t) e^{\alpha\delta} \right\}$$

$$\leq -\alpha W(t, \phi, \psi) + \eta(t) \qquad t \geq t_0 .$$

The continuity of W may be proved using arguments parallel to that of Theorem 1 of [2]. To complete the proof it remains to show that W is Lipschitzian in ϕ and ψ.

$$| W(t, \phi_1, \psi_1) - W(t, \phi_2, \psi_2) | = | \operatorname{Sup}_{\delta\geq 0} \{ \| v_{t+\delta}(t; \phi_1, \psi_1) \|_0 - H(t+\delta, t)\} e^{\alpha\delta} - \operatorname{Sup}_{\delta\geq 0} \{\| v_{t+\delta}(t; \phi_2, \psi_2) \|_0 - H(t+\delta, t)\} e^{\alpha\delta} |$$

$$\leq | \operatorname{Sup}_{\delta\geq 0} \{\| v_{t+\delta}(t; \phi_1, \psi_1) \|_0 - \| v_{t+\delta}(t; \phi_2, \psi_2) \|_0\} e^{\alpha\delta} | .$$

Using hypothesis (i) we obtain the inequality,

$$| W(t, \phi_1, \psi_1) - W(t, \phi_2, \psi_2) | \leq K(t, \tau) (\| \phi_1 - \phi_2 \|_0 + \| \psi_1 - \psi_2 \|_0), \text{ for } t \geq t_0$$

which concludes the proof.

THEOREM 2.8. *Assume that*

(i) *the* ϕ-ASI *set* $\phi = 0, \psi = 0$ *be generalized partial equi-exponential asymptotically stable and for any two solutions*

$$(v_t(t_0; \phi_1, \psi_1), w_t(t_0; \phi_1, \psi_1))$$

and

$$(v_t(t_0; \phi_2, \psi_2), w_t(t_0; \phi_2, \psi_2))$$

satisfy,

$$\| v_t(t_0; \phi_1, \psi_1) - v_t(t_0; \phi_2, \psi_2) \|_0 \leq K(t_0, \tau) (\| \phi_1 - \phi_2 \|_0 + \| \psi_1 - \psi_2) \|_0) e^{(p(t_0) - p(t))} \qquad t \geq t_0$$

where $p(t)$ *is the function of definition* 2.5;

(ii) *the function* $H(t, t_0)$ *is partially differentiable with respect to* t_0 *and*

$$\operatorname{Sup}_{\delta\geq 0} \left\{ -\frac{\partial H}{\partial t_0} (t+\delta, t) e^{(p(t+\delta) - p(t))} \right\} \leq \eta(t)$$

where $\eta \in C(R^+, R^+)$ *and* $\int_t^{t+1} \eta(s) ds \to 0$ *as* $t \to \infty$.

(iii) $f(t, \phi, \psi)$ *is linear in* ϕ, ψ *and* $p(t)$ *is continuously differentiable on* R^+. *Then there exists a Lyapunov functional* $W(t, \phi, \psi)$, *such that* (I) *and* (II) *of Theorem* 2.7 *hold and*

$$D^+ W(t, \phi, \psi) \leq -p'(t) W(t, \phi, \psi) + \eta(t)$$

for $(t, \phi, \psi) \in R^+ \times C_\rho \times C_\beta$.

Proof. Define $W(t, \phi, \psi) = \sup_{\delta \geq 0} \{\| v_{t+\delta}(t; \phi, \psi) \|_0 - H(t+\delta, t)\} \exp (p(t+\delta) - p(t))$.

Arguments parallel to that of the proof of Theorem 6.1 of [1] and Theorem 2.7 yield the required result.

Remarks. The existence of Lyapunov functional for the partial exponential asymptotic stability of the trivial solution $v = 0, w = 0$ for a linear system of equations with delay was obtained in [2]. Our Theorem 2.7 gives an analogous result for the existence of Lyapunov functional when the ϕ-ASI set $\phi = 0, \psi = 0$ of the system (1) is partially equi-exponential asymptotically stable.

Theorem 2.8 is a generalization of Theorem 2.7 and Theorem 6.1 of [1] for non-uniform stability property of an invariant set. If we take $H(t, t_0) = 0$, $K(t, \tau) = K > 0$ and f, g both linear in ϕ, ψ, then Theorem 2.7 reduces to Theorem 1 of [2]. If in addition $p(t) = \alpha t, \alpha > 0$, Theorem 2.8 also reduces to Theorem 1 of [2].

Suppose in Theorem 2.7 we define

$$W(t, \phi, \psi) = \sup_{\delta \geq 0} \{\| v_{t+\delta}(t; \phi, \psi) \|\} e^{\alpha\delta}$$

as in [2] where $f(t, \phi, \psi)$ and $g(t, \phi, \psi)$ are assumed linear in ϕ and ψ, then the proof of Theorem 1 of [2] does not carry over entirely, since all the properties of $W(t, \phi, \psi)$ in Theorem 2.7 could be derived except the upper bound of (II). This is so because

$$W(t, \phi, \psi) \leq K(t, \tau)(\| \phi \|_0 + \| \psi \|_0) + \sup_{\delta \geq 0} H(t + \sigma, t) e^{\alpha\delta}$$

and $\sup_{\delta \geq 0} H(t + \delta, t) e^{\alpha\delta}$ may not be bounded in general. For example, the function $H(t, t_0) = \int_{t_0}^{t} e^{-\alpha(t-s)} \sigma(s) \, ds$ satisfies the assumption (iii) and in particular for $\sigma(s) = e^{-\alpha s}$, $\int_{t}^{t+1} \sigma(s) \, ds \to 0$ as $t \to \infty$, but $\sup_{\delta \geq 0} H(t+\delta, t) e^{\alpha\delta} = \sup_{\delta \geq 0} e^{-\alpha t} \cdot \delta$ which does not exist. So the Lyapunov functional $W(t, \phi, \psi)$ defined above does not satisfy all the conclusions of Theorem 2.7. On the other hand, if we assume that $f(t, \phi, \psi)$ and $g(t, \phi, \psi)$ are non-linear but Lipsclitz, continuous in ϕ, ψ for constant L, we might expect our method of Theorem 6.4 of [1] to carry over without assumption (i) in Theorem 2.7. However, this is not the case, since the proof of Theorem 6.4 of [1] depends on the fact that $K(t, \tau)$ is bounded, which fails in our case. We therefore note that assumption (i) in both Theorems 2.7 and 2.8 sounds reasonable.

REFERENCES

[1] O. AKINYELE (1979) – *On partial stability of differential equations with time delay to appear*, in « Annali di mat Pura ed Appl. ».

[2] C. CORDUNEANU (1975) – *On partial stability for delay systems*, « Ann. Polon Math. », *29*, 357–362.

[3] V. LAKSHMIKANTHAM and S. LEELA (1969) – *Differential and Integral inequalities*, « Theory and Applications », Vol. I, Academic Press, New York.

[4] V. V. RUMIANTSEV and O. OZIRANER (1972) – *Lyapunov's function method in the problem of partial stability of motion*, « J. of Appl. Math. Mech. », *36*, 364–384.