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Analisi matematica. — *On the non linear vibrating rod equation* (*).

Nota I di GIOVANNI PROUSE (**), presentata (***) dal Socio L. AMERIO.

RIASSUNTO. — Si considera un'equazione non lineare che rappresenta il moto di una verga vibrante con moto trasversale senza ipotesi sull'ampiezza delle deformazioni e si associa ad essa una disequazione variazionale. Si enuncia un teorema di esistenza in grande per la soluzione di tale disequazione soddisfacente a classiche condizioni iniziali ed al contorno.

1. – Let us consider the equation, associated to the transverse motion of a vibrating rod

$$\frac{\partial^2 u}{\partial t^2} - \frac{Kl^2}{M}\frac{\partial^2 u}{\partial x^2} + \frac{Kl_0 l}{M}\frac{\partial}{\partial x}\frac{\dfrac{\partial u}{\partial x}}{\left[1+\left(\dfrac{\partial u}{\partial x}\right)^2\right]^{1/2}} + \frac{l^2 c}{M}\frac{\partial}{\partial x}\left(\frac{1}{\left[1+\left(\dfrac{\partial u}{\partial x}\right)^2\right]^{3/2}}\frac{\partial}{\partial x}\frac{\dfrac{\partial^2 u}{\partial x^2}}{\left[1+\left(\dfrac{\partial u}{\partial x}\right)^2\right]^{3/2}}\right) - f = 0 \tag{1.1}$$

where M denotes the mass of the rod, which we assume homogenous, l its length in conditions of rest, coinciding with the segment $0 \leq x \leq l$ of the x axis, $l_0 < l$ its proper length, in the absence of constraints; C and K are elastic constants, depending on the physical characteristics of the material.

Equation (1.1) has been obtained in [1] through the study of appropriate finite dimensional dynamical systems; we therefore refer to this note for a more detailed discussion of the nature and significance of the various terms appearing in (1.1).

The non linearity of (1.1) is due to the fact that no assumptions are made on the amplitude of the deformations of the rod. If we assume that $1 + (\partial u/\partial x)^2 \sim 1$, then (1.1) reduces to the classical linear equation of the vibrating rod.

We recall that $u(x, t)$ represents the displacement at the time t of the point of the rod which in its rest position has coordinate x; moreover, $f(x, t)$ represents the external force, acting perpendicularly to the x axis.

In what follows we shall assume that the rod is clamped at both ends, so that the boundary conditions are

$$u(0, t) = u(l, t) = \frac{\partial u(0, t)}{\partial x} = \frac{\partial u(l, t)}{\partial x} = 0 \qquad (0 \leq t \leq T). \tag{1.2}$$

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We shall, moreover, assign the following initial conditions

$$(1.3) \qquad u(x, 0) = z_0(x) \quad , \quad \frac{\partial u(x, 0)}{\partial t} = z_1(x) \qquad (0 \le x \le l) .$$

The aim of the present note and of the following one is to prove a *global* existence theorem of a function which, according to what will be observed below, can be assumed to represent a solution of a problem which generalises that of the vibrating rod.

The study of a global solution of (1.1), (1.2), (1.3) appears to be very difficult owing to the strong non linearity of the problem and no existence seems to have been given as yet, even for weak solutions. While, in fact, an "energy equation" can easily be established for the eventual solutions of (1.1), the a priori estimates which can be deduced from it are not sufficient to ensure the convergence of appropriate approximate solutions.

The problem formulated above in classical terms does not, however, take into account a fundamental element in the description of the physical phenomenon of the vibrating rod: the fact that when the stress reaches at some point a certain value the rod breaks and its motion is therefore no longer described by (1.1).

From this observation it follows that, for the description of the phenomenon of the vibrating rod, we can substitute to (1.1) any other relation, provided this relation coincides with (1.1) during any interval of time $0 \le t < t'$ in which the stress does not reach a breaking point value at any point of the rod; for $t \ge t'$ it can also differ from (1.1) because such equation does not, anyway, interpret the problem under consideration. As we shall see, a relation which satisfies the conditions given above is represented by a variational inequality.

A rod will break at a point $\bar{x}$ and at a time $\bar{t}$ if, denoting by T the tension, by M the bending moment and by τ the shear stress, one, at least, of the following conditions is verified

$$(1.4) \qquad |T(\bar{x}, \bar{t})| \ge N'_1 \quad , \quad |M(\bar{x}, \bar{t})| \ge N'_2 \quad , \quad |\tau(\bar{x}, \bar{t})| N'_3 ,$$

where N'_1, N'_2, N'_3 are constants depending on the material. Bearing in mind that the motion is transversal, that $\tau = - \partial M / \partial s$ and assuming that the laws of Hooke and Euler hold, relations (1.4) become

$$(1.5) \qquad \left| \frac{\partial u(\bar{x}, \bar{t})}{\partial x} \right| \ge N'_1 \quad , \quad \left| \frac{\dfrac{\partial^2 u}{\partial x^2}}{\left[1 + \left(\dfrac{\partial u}{\partial x}\right)^2\right]^{3/2}} \right| \ge N'_2 ,$$

$$\left| \frac{\dfrac{\partial^3 u}{\partial x^3}}{\left[1 + \left(\dfrac{\partial u}{\partial x}\right)^2\right]^2} - \frac{3 \dfrac{\partial u}{\partial x} \left(\dfrac{\partial^2 u}{\partial x^2}\right)^2}{\left[1 + \left(\dfrac{\partial u}{\partial x}\right)^2\right]^3} \right| \ge N'_3 .$$

Let N_1'', N_2'', N_3'' be appropriate constants (depending on N_1', N_2', N_3'); it can easily be shown that, if one, at least, of the relations

$$(1.6) \qquad \left|\frac{\partial u(\bar{x},\bar{t})}{\partial x}\right| = N_1'' \quad , \quad \left|\frac{\partial^2 u(\bar{x},\bar{t})}{\partial x^2}\right| = N_2'' \quad , \quad \left|\frac{\partial^3 u(\bar{x},\bar{t})}{\partial x^3}\right| = N_3''$$

holds at the point $(\bar{x},\bar{t})$, then necessarily one at least of (1.5) holds at the same point and consequently the rod breaks. It appears therefore natural to take into account (1.6) when studying the problem of the vibrating rod. More precisely, equation (1.1) may have a physical interpretation only provided that

$$(1.7) \qquad \left|\frac{\partial u}{\partial x}\right| < N_1'' \quad , \quad \left|\frac{\partial^2 u}{\partial x^2}\right| < N_2'' \quad , \quad \left|\frac{\partial^3 u}{\partial x^3}\right| < N_3'' .$$

2. – We now introduce an equation which generalises (1.1) and which we shall therefore consider in the sequel in place of (1.1). Observe that the Hamiltonian function associated to (1.1) has the form

$$(2.1) \qquad \mathscr{H}(t) = \frac{M}{l}\int_0^l \left[\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 - H\left(\frac{\partial u}{\partial x}\right) - \frac{1}{2}\left(\frac{\partial}{\partial x} G\left(\frac{\partial u}{\partial x}\right)\right)^2\right] dx$$

with

$$(2.2) \qquad H(\eta) = \int_0^\eta h(\xi)\, d\xi \quad , \quad G(\eta) = \int_0^\eta g(\xi)\, d\xi$$

and

$$(2.3) \qquad h(\xi) = \frac{Kl^2}{M}\xi - \frac{Kl_0 l}{M}\frac{\xi}{(1+\xi^2)^{1/2}} \quad , \quad g(\xi) = \sqrt{\frac{cl^2}{2M}}\frac{1}{(1+\xi^2)^{3/2}} .$$

Hence, if we neglect (2.3) and assume that $h(\xi)$, $g(\xi)$ are arbitrary continuous functions with $g \in C^1$, from the Hamiltonian (2.1) we decuce the equation

$$(2.4) \qquad \int_{t_1}^{t_2}\int_0^l \left[\frac{\partial u}{\partial t}\frac{\partial \varphi}{\partial t} - h\left(\frac{\partial u}{\partial x}\right)\frac{\partial \varphi}{\partial x} - \right.$$
$$\left. - g\left(\frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x^2}\frac{\partial}{\partial x}\left(g\left(\frac{\partial u}{\partial x}\right)\frac{\partial \varphi}{\partial x}\right) + f\varphi\right] dx\, dt = 0$$

where $\varphi(x,t)$ is an arbitrary function with

$$\varphi(x,t_1) = \varphi(x,t_2) = \varphi(0,t) = \varphi(l,t) = \varphi_x(0,t) = \varphi_x(l,t) = 0 .$$

Equation (2.4) represents the variational formulation of the equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x} h \left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x} \left[g \left(\frac{\partial u}{\partial x}\right) \frac{\partial}{\partial x} \left(g \left(\frac{\partial u}{\partial x}\right) \frac{\partial^2 u}{\partial x^2} \right) \right] - f = 0 \tag{2.5}$$

which, when (2.3) hold, reduces to (1.1).

For the sake of generality, we shall, from now on, consider (2.5) in place of (1.1), associating to it the initial and boundary conditions (1.2) and (1.3) and the limitations (1.7); the corresponding problem will be called the *generalised vibrating rod problem.*

3. – Let us now construct an inequality which, according to what has been said in the preceding paragraphs, can be associated to the generalised vibrating rod problem.

Let K be the convex set

$$\mathrm{K} = \left\{ v(t) \in \mathrm{C}^1(0, \mathrm{T}; \mathrm{H}^{3,\infty}) : \left|\frac{\partial v}{\partial x}\right| \leq \mathrm{N}_1'' \quad , \quad \left|\frac{\partial^2 v}{\partial x^2}\right| \leq \mathrm{N}_2'' , \right. \\ \left. \left|\frac{\partial^3 v}{\partial x^3}\right| \leq \mathrm{N}_3'' \quad \text{a.e. in} \quad \mathrm{Q} = (0, l) \times (0, \mathrm{T}) \right\} \tag{3.1}$$

and denote by K_0 its closure in $\mathrm{H}^1(0, \mathrm{T}; \mathrm{L}^2)$ (1); it is, obviously,

$$\mathrm{K}_0 = \left\{ v(t) \in \mathrm{H}^1(0, \mathrm{T}; \mathrm{L}^2) : \left|\frac{\partial v}{\partial x}\right| \leq \mathrm{N}_1'' \quad , \quad \left|\frac{\partial^2 v}{\partial x^2}\right| \leq \mathrm{N}_2'' , \right. \\ \left. \left|\frac{\partial^3 v}{\partial x^3}\right| \leq \mathrm{N}_3'' \quad \text{a.e. in} \quad \mathrm{Q} \right\} . \tag{3.2}$$

Setting

$$\mathrm{A}u = - \frac{\partial}{\partial x} h \left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x} \left[g \left(\frac{\partial u}{\partial x}\right) \frac{\partial}{\partial x} \left(g \left(\frac{\partial u}{\partial x}\right) \frac{\partial^2 u}{\partial x^2} \right) \right] , \tag{3.3}$$

assuming that $u(t) \in \mathrm{K}_0 \cap \mathrm{L}^2(0, \mathrm{T}; \mathrm{H}_0^2)$, $\frac{\partial^2 u}{\partial t^2} \in \mathrm{L}^2(\mathrm{Q})$, $\mathrm{A}u \in \mathrm{L}^2(\mathrm{Q})$ and denoting by φ any function $\in \mathrm{K}_0$, with $\frac{\partial \varphi}{\partial t} \in \mathrm{L}^2(\mathrm{Q})$, we consider the inequality

$$\int_{\mathrm{Q}_t} \left(\frac{\partial^2 u}{\partial t^2} + \mathrm{A}u - f \right) \left(\frac{\partial \varphi}{\partial t} - \frac{\partial u}{\partial t} \right) d\mathrm{Q}_t \geq 0 \tag{3.4}$$

$$(\mathrm{Q}_t = (0, l) \times (0, t), \ 0 < t \leq \mathrm{T}) .$$

(1) From now on we shall write $u(t) = \{u(x, t); x \in (0, l)\}$, $\mathrm{L}^2 = \mathrm{L}^2(0, l)$, $\mathrm{H}^s = \mathrm{H}^s(0, l)$.

It is well known that, if (1.7) hold in an open set $Q' \subseteq Q$, then

$$\frac{\partial^2 u}{\partial t^2} + Au - f = 0 \qquad \text{a.e. in } Q'. \tag{3.5}$$

Hence, it is natural, by what has been previously observed, to associate to the generalised vibrating rod problem the inequality (3.4) with the initial conditions (1.3) (the boundary conditions are accounted for by the assumption that $u(t) \in L^2(0, T; H_0^2))$.

Let us now give a weak formulation of (3.4). Observe, to begin with, that, setting

$$u'(t) = \left\{\frac{\partial u(x,t)}{\partial t} \; ; \; x \in (0, l)\right\} \quad , \quad Du(t) = \left\{\frac{\partial u(x,t)}{\partial x} \; ; \; x \in (0, l)\right\},$$

we have, bearing in mind (1.3),

$$\int_0^t (u'' + Au, u')_{L^2} \, d\eta = \tfrac{1}{2} \| u'(t) \|_{L^2}^2 + \tag{3.6}$$

$$+ (H(Du(t)), 1)_{L^2} + \tfrac{1}{2} \| G(Du(t)) \|_{H_0^1}^2 -$$

$$- \tfrac{1}{2} \| z_1 \|_{L^2}^2 - (H(Dz_0), 1)_{L^2} - \tfrac{1}{2} \| G(Dz_0) \|_{H_0^1}^2 .$$

Moreover, $\forall \psi \in L^2(0, T; H_0^2)$,

$$\int_0^t (Au, \psi)_{L^2} \, d\eta = \tag{3.7}$$

$$= \int_0^t [(h(Du), D\psi)_{L^2} + (g(Du) D^2 u, D(g(Du) D\psi))_{L^2}] \, d\eta = \int_0^t a(u, \psi) \, d\eta$$

Substituting (3.6), (3.7) into (3.4) we obtain, assuming that $\varphi(t) \in K_0$, $\varphi'(t) \in L^2(0, T; H_0^2)$, $\varphi''(t) \in L^2(0, T; L^2)$,

$$- \tfrac{1}{2} \| u'(t) \|_{L^2}^2 - (H(Du(t)), 1)_{L^2} - \tfrac{1}{2} \| G(Du(t)) \|_{H_0^1}^2 + (u'(t), \varphi'(t))_{L^2} + \tag{3.8}$$

$$+ \int_0^t [a(u, \varphi') - (u', \varphi'')_{L^2} - (f, \varphi' - u')_{L^2}] \, d\eta \geq$$

$$\geq - \tfrac{1}{2} \| z_1 \|_{L^2}^2 - (H(Dz_0), 1)_{L^2} - \tfrac{1}{2} \| G(Dz_0) \|_{H_0^1}^2 + (z_1, \varphi'(0))_{L^2} .$$

We shall then say that $u(t)$ is a *weak solution* in $[0, T]$ of (3.4), (1.2), (1.3) if:

a') $u(t) \in K_0 \cap L^2(0, T; H_0^2)$, $u(0) = z_0$;

b') *$u(t)$ satisfies (3.8) a.e. in $[0, T]$ $\forall \varphi(t) \in K_0$ with $\varphi'(t) \in L^2(0, T; H_0^2)$,* $\varphi''(t) \in L^2(0, T; L^2)$.

In the subsequent note II we shall prove the following theorem.

THEOREM I. – *Assume that:*

(*i*) $f(t) \in L^2(o, T; L^2)$;

(*ii*) $z_0 \in H_0^2 \cap H^{3,\infty}$, *with* $|Dz_0| \leq N_1''$, $|D^2 z_0| \leq N_2''$, $|D^3 z_0| \leq N_3''$ *a.e. in* (o, l); $z_1 \in L^2$;

(*iii*) $h(\xi) \in C^0(-\infty, \infty)$, $g(\xi) \in C^1(-\infty, \infty)$ *and there exist two positive constants*, M' *and* c', *such that*

$$-M' \leq (H(v), 1)_{L^2} + \tfrac{1}{2} \| G(v) \|_{H_0^1}^2 \leq c' \| v \|_{H_0^1}^2 \qquad \forall v \in H_0^1 .$$

Then there exists at least one function $u(t)$ *satisfying* a'), *b'*).

OBSERVATION I. – Let $\tilde{K}$ be a convex set defined by

$$(3.9) \qquad \tilde{K} = \left\{ v(t) \in C^0(o, T; H^{3,\infty}) : \left| \int_0^t \frac{\partial v}{\partial x} d\eta + Dv(o) \right| \leq N_1'' , \right.$$

$$\left. \left| \int_0^t \frac{\partial^2 v}{\partial x^2} d\eta + D^2 v(o) \right| \leq N_2'' , \left| \int_0^t \frac{\partial^3 v}{\partial x^3} d\eta + D^3 v(o) \right| \leq N_3'' \quad \text{a.e. in } Q \right\}$$

and denote by $\tilde{K}_0$ the closure of $\tilde{K}$ in $L^2(o, T; L^2)$. Let, moreover, $u_0 \in H_0^2$, with $|Du_0| \leq N_1''$, $|D^2 u_0| \leq N_2''$, $|D^3 u_0| \leq N_3''$. Then, $\forall u(t)$ such that $u(o) = u_0$,

$$(3.10) \qquad u(t) \in K_0 \Longleftrightarrow u'(t) \in \tilde{K}_0 .$$

Observe, in fact, that, by the definitions given, it is obviously, if $u(o) = u_0$,

$$(3.11) \qquad u(t) \in K \Longleftrightarrow u'(t) \in \tilde{K} .$$

Let now $u(t) \in K_0$, $u(o) = u_0$ and $\{u_n(t)\}$ be a sequence such that

$$(3.12) \qquad u_n(t) \in K \quad , \quad \lim_{n \to \infty} u_n(t) \underset{H^1(o,T;L^2)}{=} u(t) \quad , \quad u_n(o) = u_0 .$$

We have then, by (3.11), $u_n'(t) \in \tilde{K}$ and, obviously,

$$(3.13) \qquad \lim_{n \to \infty} u_n'(t) \underset{L^2(o,T;L^2)}{=} u'(t) .$$

Hence, by (3.13), $u'(t) \in \tilde{K}_0$ and we have therefore proved that, if $u(o) = u_0$,

$$(3.14) \qquad u(t) \in K_0 \Rightarrow u'(t) \in \tilde{K}_0 .$$

Assume now that $u'(t) \in \tilde{K}_0$, $u(o) = u_0$ and let $\{u_n(t)\}$ be a sequence such that

$$(3.16) \qquad u_n'(t) \in \tilde{K}_0 \quad , \quad \lim_{n \to \infty} u_n'(t) \underset{L^2(o,T;L^2)}{=} u'(t) \quad , \quad u_n(o) = u_0$$

By (3.11) we have $u_n(t) \in K$ and, moreover, by (3.15).

$$\lim_{n \to \infty} u_n(t) \underset{H^1(0,T;L^2)}{=} u(t) . \tag{3.17}$$

Hence $u(t) \in K_0$, which proves that, if $u(0) = u_0$,

$$u'(t) \in \tilde{K}_0 \Rightarrow u(t) \in K_0 .$$

OBSERVATION 2. – By what has been proved in observation 1, in the definition of weak solution, condition a') can be substituted by the following:

a') $u(t) \in L^2(0, T; H_0^2)$, $u'(t) \in \tilde{K}_0$, $u(0) = z_0$, *with* z_0 *satisfying condition* (ii) *of Theorem 1.*

OBSERVATION 3. – Condition (iii) of Theorem 1 is verified if the functions $h(\xi), g(\xi)$ are defined by (2.3), i.e. when (2.5) reduces to the equation (1.1) of the vibrating rod. In this case, denoting by σ the embedding constant of H_0^2 in H_0^1, we may choose $M' = \frac{5 K l^2}{8M}$, $c' = \frac{l^2}{2M}(C + K\sigma^2)$.

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