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A variational approach to the polar decomposition theorem

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RENDICONTI

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *A variational approach to the polar decomposition theorem.* Nota di LUIZ CARLOS MARTINS e PAOLO PODIO-GUIDUGLI (*), presentata (**) dal Corrisp. G. GRIOLI.

RIASSUNTO. — In cinematica dei continui si ricorre al teorema di decomposizione polare dell'algebra lineare per risolvere localmente una deformazione nella successione di una deformazione pura e di una rotazione; Grioli [4] ha mostrato che tale rotazione è l'approssimazione rigida di minima distanza dalla deformazione data. In questo articolo si mostra come una caratterizzazione variazionale del tipo di Grioli conduca naturalmente a stabilire il teorema di decomposizione polare.

1. INTRODUCTION

Let Lin denote the space of all linear transformations of a finite-dimensional real vector space $\mathscr{V}$ into itself. Further, let Sym be the subspace of the symmetric elements of Lin, and let Orth be the full orthogonal group. Finally, let Inv and Pos be the collections of invertible and positive elements of Lin and Sym, respectively.

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The Polar Decomposition Theorem [1] is the following proposition:

Given $F \in$ Lin, *there exist* $R \in$ Orth *and two (uniquely determined) elements* $U, V \in$ Pos *such that*

$$F = RU = VR. \tag{1.1}$$

Moreover, if $F \in$ Inv, *the orthogonal factor* R *in the decompositions* (1) *of* F *is uniquely determined.*

Now, consider the following problem:

$$\textit{Given } F \in \text{Lin}, \textit{ find } \underset{Q \in \text{Orth}}{\text{minimum}} \, | F - Q | . \tag{1.2}$$

In justification of our title, we will show hereafter that solving the above variational problem leads one to establish the polar decomposition theorem. Thus, in particular, the orthogonal factor R supplies the best approximation of a given $F \in$ Inv in the sense that R is the point of minimal distance of F from Orth.

The last remark provides the link with the application which motivated our work. In fact, in Continuum Mechanics it is customary to apply the polar decomposition theorem to resolve locally a deformation into the succession of either a pure stretch and a rotation (with gradients U and R, respectively) or a rotation and a pure stretch (with gradients R and V).

Many years ago, G. Grioli [4] posed the problem of finding the rigid deformation whose difference (in terms of the L^2 norm of displacements) from a given homogeneous deformation is the least possible. Grioli was able to prove that the best rigid deformation is determined by the orthogonal factor in the polar decomposition of the gradient of the given deformation.

On reconsidering Grioli's paper, we noticed that exploiting the polar decomposition theorem is not necessary to arrive at his variational characterization of the rotation. Moreover, Grioli's problem can be generalized as in (2), and solved yielding a new and more elementary proof of the polar decomposition theorem.

2. A Hint

Let us endow Lin with the usual inner product:

$$A \cdot B = tr\,(AB^T) .$$

(1) This theorem subtends the general theory of strain created by A. Cauchy between 1823 and 1841, but it was not stated explicitly by him; as an algebraic result, it seems apparently originated in a paper by J. Finger appeared in 1892. These, and many other, historical informations are found in Section 25–27 of [1] and Section 43 of [2]; the archetypal modern proof is found in Section 83 of [3].

where A , B are any two elements of Lin, B^T is the transpose of B and *tr* is the trace functional. Accordingly, let us define the magnitude $|A|$ of $A \in$ Lin as

$$|A| = (A \cdot A)^{\frac{1}{2}} ;$$

in particular, if $Q \in$ Orth,

$$|Q| = n^{\frac{1}{2}} ,$$

where $n = \dim \mathscr{V}$. For any $F \in$ Lin and $Q \in$ Orth we then have

$$|F - Q|^2 = (F - Q) \cdot (F - Q) = |F|^2 + n - 2 F \cdot Q.$$

Thus, problem (1.2) is equivalent to the following problem:

(2.1) *Given* $F \in$ Lin, *find* $\underset{Q \in \mathrm{Orth}}{\text{maximum}}\ F \cdot Q$.

With a view towards giving a hint which will prove crucial in the sequel, we show first how Grioli [4] solved the above problem in the special case (of interest in the application he aimed at) in which F is given in Inv^+ and Q is searched for in Orth^+ (here Inv^+ and Orth^+ are the collections of all elements with positive determinant of Inv and Orth, respectively).

Taking the polar decomposition theorem for granted, one has

$$F = VR \quad \text{with} \quad V \in \mathrm{Pos}^+ \qquad \text{and} \quad R \in \mathrm{Orth}^+ \text{ (2)}.$$

Further, one writes

$$F \cdot Q = VR \cdot Q = V \cdot QR^T.$$

In view of a lemma to be stated later in greater generality,

$$V \cdot H \leq V \cdot I \quad , \quad \forall H \in \mathrm{Orth}^+,$$

with equality holding only if $H = I$, the identity transformation of $\mathscr{V}$ into itself. Thus, one concludes with Grioli that, under the present circumstances, the unique solution of problem (1) is encountered when $QR^T = I$, i.e., $Q = R$.

Secondly, we approach Grioli's problem via differential calculus on manifolds. Then, a necessary condition for the functional $Q \mapsto F \cdot Q$ to attain its maximum at $R \in \mathrm{Orth}^+$ is that any of its directional derivatives vanish at R , i.e.,

(2.2) $$F \cdot WR = 0 \quad , \quad \forall W \in \mathrm{Skw} \text{ (3)}.$$

In turn, (2) is equivalent to

(2.3) $$FR^T \in \mathrm{Sym} .$$

(2) Pos^+ denotes the collection of all strictly positive elements of Pos.

(3) Skw is the subspace of the skew-symmetric elements on Lin: as is well known, Skw Q is the tangent space to Orth^+ at the point Q.

We leave it to the reader the easy task of recovering Grioli's result by iterate use of the polar decomposition theorem. Rather, we point out the hint offered by (3):

If $|F - Q|$ *has to be minimized over* Orth, *then any point of minimum* R *has to be such as to deliver a polar decomposition of* F *in the form*

$$F = VR \,, \quad \textit{with} \quad V \in \mathrm{Sym} \,. \tag{2.4}$$

3. Two Lemmata

We have seen in the previous section that there is reason for undertaking the study of the set of maximizers of the functional defined, for any $A \in \mathrm{Lin} - \{0\}$, as

$$\lambda_A : \mathrm{Orth} \to \mathbf{R} \quad , \quad \lambda_A(H) = A \cdot H \,. \tag{3.1}$$

The following lemmata, which make explicit the relations of maximum properties of λ_A to positiveness of A, will play a central role in solving problem (2.1).

LEMMA 1. *Let* λ_A *as above have maximal value at* $H = I$. *Then,* A *is positive.*

Proof. Under the hypothesis,

$$A \cdot H \leq A \cdot I \quad , \quad \forall H \in \mathrm{Orth} \,. \tag{3.2}$$

It the follows from (2.3) that A is symmetric. To show that A is positive, assume that there exists $v \in \mathscr{V}$ such that $v \cdot Av < 0$ and $|v| = 1$, and choose

$$H = I - 2\, v \otimes v \; {}^{(4)}.$$

Then,

$$A \cdot H = A \cdot I - 2\, v \cdot Av > A \cdot I \,,$$

and we have a contradiction. □

LEMMA 2. *Let* A *be positive*; *denote the range of* A *by* $\mathscr{R}(A)$. *Then,* λ_A *as in* (1) *has maximal value at* $H = I$, *and* $\lambda_A(H) = \lambda_A(I)$ *implies* $(H - I)|_{\mathscr{R}(A)} = 0$ [5].

(4) When $\mathscr{V}$ is made an inner product space, $v \otimes v$ is the element defined by $(v \otimes v)\, u = (v \cdot u) v$, $\forall u \in \mathscr{V}$. The inner products of $\mathscr{V}$ and Lin match in the following sense: $v \cdot Av = A \cdot v \otimes v$, $\forall A \in \mathrm{Lin}$ and $\forall v \in \mathscr{V}$.

(5) After the completion of this work we became aware of the existence of a paper by J. von Neumann where the assertion of Lemma 1 is proved in a more general from (cfr. [5], Lemma 7). Also, part of the assertion of Lemma 2, namely, the implication:

$$(A \in \mathrm{Pos}) \Rightarrow (\lambda_A \text{ has maximal value at } I)$$

is a corollary of Theorem I of [5]. However, our proofs are not only different, but also more simple and direct than Von Neumann's (and they could easily be adapted to cover the complex case as well).

Proof. Note that, as A is symmetric,

$$(3.3)\quad \mathrm{A}\cdot(\mathrm{H}-\mathrm{I}) = -\tfrac{1}{2}\mathrm{A}\cdot(\mathrm{H}-\mathrm{I})(\mathrm{H}-\mathrm{I})^{\mathrm{T}} = -\tfrac{1}{2}(\mathrm{H}-\mathrm{I})^{\mathrm{T}}\mathrm{A}(\mathrm{H}-\mathrm{I})\cdot\mathrm{I}\,, \quad \forall \mathrm{H}\in \mathrm{Orth}\,.$$

Moreover, as A is positive,

$$(3.4)\qquad (\mathrm{H}-\mathrm{I})^{\mathrm{T}}\mathrm{A}(\mathrm{H}-\mathrm{I})\in \mathrm{Pos}\,,\quad \forall \mathrm{H}\in \mathrm{Orth}\,,$$

and we conclude that, under the present circumstances, (2) holds, i.e., λ_{A} has maximal value at $\mathrm{H} = \mathrm{I}$.

Next, we show that

$$(3.5)\qquad \mathrm{A}\cdot\mathrm{H} = \mathrm{A}\cdot\mathrm{I} \quad \text{implies} \quad (\mathrm{H}-\mathrm{I})\,|_{\mathscr{R}(\mathrm{A})} = 0\,.$$

Indeed, in view of (3) and (4), equation $(5)_1$ is equivalent to

$$(3.6)\qquad (\mathrm{H}-\mathrm{I})\,v\cdot\mathrm{A}(\mathrm{H}-\mathrm{I})\,v = 0 \quad , \quad \forall v\in \mathscr{V}\,,$$

or rather

$$(3.7)\qquad \mathrm{A}(\mathrm{H}-\mathrm{I})\,v = 0 \quad , \quad \forall v\in \mathscr{V}\,.$$

To prove the last equivalence, assume that there exist $v\in\mathscr{V}$ such that (6) holds and $\mathrm{A}(\mathrm{H}-\mathrm{I})\,v = \mathrm{A}u \neq 0$. Choose $\alpha\in\mathbf{R}$ arbitrarily, and put

$$w = \alpha u + \mathrm{A}u\,.$$

Then,

$$(3.8)\qquad 0 \le w\cdot\mathrm{A}w = (\alpha u + \mathrm{A}u)\cdot(\alpha\,\mathrm{A}u + \mathrm{AA}u) = 2\,\alpha\,\mathrm{A}u\cdot\mathrm{A}u + \mathrm{A}u\cdot\mathrm{AA}u\,, \quad \forall\alpha\in\mathbf{R}\,.$$

But, for $\alpha < 0$ appropriately chosen,

$$2\,\alpha\,\mathrm{A}u\cdot\mathrm{A}u + \mathrm{A}u\cdot\mathrm{AA}u < 0\,,$$

which contradicts (8) and establishes (7). In turn, (7) implies that

$$0 = \mathrm{A}(\mathrm{H}-\mathrm{I})\,\mathrm{A}u\cdot u = (\mathrm{H}-\mathrm{I})\,\mathrm{A}u\cdot\mathrm{A}u \quad , \quad \forall u\in\mathscr{V}\,,$$

or rather

$$0 = (\mathrm{H}-\mathrm{I})\,v\cdot v \quad , \quad \forall v\in\mathscr{R}(\mathrm{A})\,. \qquad \square$$

4. THE POLAR DECOMPOSITION THEOREM

For convenience, we begin by restating problem (2.1):

$$(4.1)\qquad \textit{Given } \mathrm{F}\in \mathrm{Lin}\,,\ \textit{find}\ \underset{\mathrm{Q}\in\mathrm{Orth}}{\text{maximum}}\ \mathrm{F}\cdot\mathrm{Q}\,.$$

As Orth is a compact set, the problem makes sense and has a non-empty solution set $\mathscr{S}$. If $\mathrm{R}\in\mathscr{S}$,

$$\mathrm{F}\cdot\mathrm{Q} \le \mathrm{F}\cdot\mathrm{R} \quad , \quad \forall \mathrm{Q}\in \mathrm{Orth}\,,$$

or rather

$$FR^T \cdot QR^T \leq FR^T \cdot I \quad , \quad \forall Q \in \text{Orth} ,$$

i.e., the functional λ_A defined as in (3.1), with $A = FR^T$ and $H = QR^T$, has maximal value at $H = I$. In view of Lemma 1, we then conclude that

$$V = FR^T \in \text{Pos} , \tag{4.2}$$

and the existence of a polar decomposition is established for an arbitrary element F of Lin.

Next, we show that V is uniquely determined. Suppose that $V, \bar{V} \in \text{Pos}$ and that

$$F = VR = \overline{V}\overline{R} \quad \text{for some} \quad R, \bar{R} \in \text{Orth} . \tag{4.3}$$

If $Q \in \text{Orth}$, $(3)_2$ implies that

$$V \cdot QR^T = \bar{V} \cdot Q\bar{R}^T .$$

Now, choose Q such that the functional λ_A, with $A = V$ and $H = QR^T$, has maximal value. By appealing to Lemma 2, we have that both

$$(QR^T - I)\,|_{\mathscr{R}(V)} = 0 \tag{4.4}$$

and

$$(Q\bar{R}^T - I)\,|_{\mathscr{R}(\bar{V})} = 0 . \tag{4.5}$$

It follows from (4) and (5), respectively, that

$$QR^T V = V \quad \text{and} \quad Q\bar{R}^T \bar{V} = \bar{V} . \tag{4.6}$$

By (6) and $(3)_2$,

$$Q^T V = R^T V = \bar{R}^T \bar{V} = Q^T \bar{V} \;,$$

which implies

$$V = \bar{V} . \tag{4.7}$$

Thus, solving the variational problem (1.2) in its equivalent form (1) has led us to establish the polar decomposition $(1.1)_2$. As a bonus, by use of (6) in the light of (7), we can make precise the structure of the solution set $\mathscr{S}$ of problem (1):

$$\text{If } R \in \mathscr{S}, \text{ then } (\bar{R}R^T - I)\,|_{\mathscr{R}(V)} = 0 , \quad \text{for any } \bar{R} \in \mathscr{S} .$$

To complete our task, we observe that decomposition $(1.1)_1$ is arrived at by applying the same procedure to F^T. Finally, the special case when $F \in \text{Inv}$ is easily dealt with once one notices that, under the hypothesis, V is invertible, and hence $\mathscr{S}$ reduces to a singleton. □

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