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Comparison between optical and totally geodesic coordinates in general relativity in connection with the first two Cauchy-Green tensors

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Teorie relativistiche. — *Comparison between optical and totally geodesic coordinates in general relativity in connection with the first two Cauchy-Green tensors* (*). Nota di ANTONIO CLAUDIO GRIOLI e RAFFAELE BALLI, presentata (**) dal Socio D. GRAFFI.

RIASSUNTO. — A. Bressan fonda la sua teoria relativistica dei materiali di ordine n su una relativizzazione del gradiente n-simo della posizione ottenuta mediante derivazione su una superficie totalmente geodetica e tempo ortogonale.

D'altra parte Lianis, nella sua teoria di elasticità relativistica, introduce una relativizzazione del gradiente primo di posizione, differente almeno concettualmente da quello di Bressan, espresso in coordinate ottiche mediante derivazione della posizione sulla ipersuperficie $\overset{\circ}{\zeta}{}^0 = \text{cost.}$

In questo lavoro calcoliamo l'espressione della derivata di un arbitrario doppio tensore sulla ipersuperficie $\overset{\circ}{\zeta} = \text{cost.}$ e la chiamiamo derivata ottica, quindi determiniamo il gradiente secondo della posizione in coordinate ottiche, infine mostriamo che, mentre le espressioni dei gradienti primi di Bressan e Lianis coincidono, i gradienti secondi differiscono per termini dipendenti dall'accelerazione. Pertanto le due teorie coincidono soltanto per quanto riguarda i materiali del primo ordine.

1. INTRODUCTION

A. Bressan (see [1]) has introduced, for developing his theory about materials on n-th order, a sort of relativistic n-th gradient of position, obtained by total derivation of the position on the totally geodesic and time-orthogonal hypersurface. The way of calculating this derivative for every value of n was indicated in [2].

On the other side, for his theory of relativistic elasticity, Lianis has introduced (see [3]) a relativistic first gradient of position, different from Bressan's at least conceptually, expressed in optical coordinates by total derivation of the position on the hypersurface $\overset{\circ}{\xi}{}^0 = \text{const.}$

In this paper we calculate the expression of the derivative of an arbitrary double tensor on the hypersurface $\overset{\circ}{\xi}{}^0 = \text{const.}$, and we call it optical derivative. Afterwards we determine the second gradient of position in optical co-ordinates. Finally we show that the expressions for the first gradient determined by Bressan and Lianis coincide while for the second gradient they differ in terms depending on the acceleration. Therefore the two theories coincide only for first-order materials.

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2. THE OPTICAL DERIVATIVE

We denote by S_4 a dinamically possible determination of the space-time in general (or special) relativity and by (x^ρ) an admissible system of co-ordinates; furthermore we assume the signature $-+++$ with $g_{00} < 0$ for the metric tensor $g_{\rho\sigma}$ of S_4.

We consider a continuum system C moving in S_4 and we denote by S_3^* the intersection of the world tube described by C with a spatial cross-section of S_4. In this section we put (1):

$$y^L = \delta^L_\rho \, x^\rho \qquad \overset{*}{a}_{LM} = \overset{\perp}{g}_{LM} \qquad (\text{on } S_3^*) \tag{1}$$

with:

$$\overset{\perp}{g}_{\rho\sigma} = g_{\rho\sigma} + u_\rho \, u_\sigma \tag{2}$$

Afterwards for the sake of semplicity, we identify a particle with the triple y^L of its co-ordinates in the reference configuration; we also denote by W_{y^L} the world-line of y^L, and by P an arbitrary event point on $W_{y^L+dy^L}$, the world-line of the particle $y^L + dy^L$.

We consider, moreover, along W_{y^L}, an orthonormal tetrad $\overset{\circ}{\Lambda}{}^\rho_\sigma$, in which three of the unit vectors $\overset{\circ}{\Lambda}{}^\rho_m$ are spacelike and satisfy the Fermi-Walker transport equation (2), and the last one, $\overset{\circ}{\Lambda}{}^\rho_0$, corresponding to the identification of the 4-velocity of matter:

$$\overset{\circ}{\Lambda}{}^\rho_0 = u^\rho \, . \tag{3}$$

We denote by P_1 that event point on W_{y^L} from which an observer fixed on y^L must give out a light signal to have it arrive at P. Furthermore we denote by P_2 the event point to which the reflected signal from P comes back to the observer.

We call s_1 and s_2 the values for P_1 and P_2 respectivels, of the proper time on W_{y^L}, furthermore we put:

$$\tau(P) = \frac{s_2 + s_1}{2} \tag{4}$$

$$l(P) = \frac{s_2 - s_1}{2} \, . \tag{5}$$

(1) Afterwards we state that the Greek indices range from 0 to 3, while the Latin indices range from 1 to 3.

(2) See for example [5].

The optical co-ordinates of P are defined, as is known (see [3]), by:

$$\mathring{\xi}^0 = \tau(\mathrm{P})$$

$$\mathring{\xi}^l = l(\mathrm{P})\,\mu_\rho(\bar{s})\,\mathring{\Lambda}^{\rho l}(\bar{s})$$

where μ_ρ is the unit vector of the geodesic through P, ortogonal with W_{y^L}, and where $\bar{s}$ is the value of the proper-time, for the intersection point of that geodesic with W_{y^L}.

The link between the optical co-ordinates and the (x^ρ) was calculated in [3] by Lianis. Now we denote by $T^{\alpha_1,\dots,\alpha_r}_{\beta_1,\dots,\beta_s}$ a double tensor connected with the points x^ρ of S_4 and y^L of S_3^* that are mutually related by the equations of the motion of C:

$$x^\rho = x^\rho(t, y^L). \tag{6}$$

We call *optical derivative* of $T^{\alpha_1,\dots,\alpha_r}_{\beta_1,\dots,\beta_s}$ with respect to y^A at the event point E on W_{y^L}, and we denote it by $T^{\alpha_1,\dots,\alpha_r}_{\beta_1,\dots,\beta_s \Vert A}$, the total covariant derivative of $T^{\alpha_1,\dots,\alpha_r}_{\beta_1,\dots,\beta_s}$ with respect to y^A carried out on the hypersurface Γ for E of equation (3):

$$\mathring{\xi}^0(x^\rho) = 0. \tag{7}$$

Therefore this derivative has to be carryed out by total derivation of $T = T^{\alpha_1,\dots,\alpha_r}_{\beta_1,\dots,\beta_s}[x^\rho[t(\tilde{t}, y^L), y^L]]$ with respect to y^A for a choice $\tilde{t}$ of the time parameter $\tilde{t} = \tilde{t}(t, y^L)$ in (6), such that the equation $\tilde{t}(x^\rho) = 0$ represents Γ (4). We have therefore:

$$T^{\alpha_1,\dots,\alpha_r}_{\beta_1,\dots,\beta_s \Vert A} = [T^{\alpha_1,\dots,\alpha_r}_{\beta_1,\dots,\beta_s}(x^\rho(t(\tilde{t}, y^L), y^L))]_{;A}. \tag{8}$$

The same definition is valid for the derivatives of higher order:

$$T^{\alpha_1,\dots,\alpha_r}_{\beta_1,\dots,\beta_s \Vert A_1,\dots,A_n} = [T^{\alpha_1,\dots,\alpha_r}_{\beta_1,\dots,\beta_s\, t=t(\tilde{t},y^L)}]_{;A_1,\dots,A_n}. \tag{9}$$

We limit ourselves to calculate those derivatives up to $n = 2$. If we identify t with y^0, we have:

$$T^{\alpha_1,\dots,\alpha_r}_{\beta_1,\dots,\beta_s \Vert A} = T^{\alpha_1,\dots,\alpha_r}_{\beta_1,\dots,\beta_s/\rho}(x^\rho_A + x^\rho_O\, y^0_A) + T^{\alpha_1,\dots,\alpha_r}_{\beta_1,\dots,\beta_s/A} \tag{10}$$

$$\begin{aligned} T^{\alpha_1,\dots,\alpha_r}_{\beta_1,\dots,\beta_s \Vert AB} &= T^{\alpha_\Gamma,\dots,\alpha_r}_{\beta_1,\dots,\beta_s/\rho\tau}(x^\rho_A + x^\rho_O\, y^0_A)(x^\tau_B + x^\tau_O\, y^0_B) + \\ &+ T^{\alpha_1,\dots,\alpha_r}_{\beta_1,\dots,\beta_s/\rho}(x^\rho_{/AB} + x^\rho_{/AO}\, y^0_B + x^\rho_{/OB}\, y^0_A + x^\rho_{/OO}\, y^0_A\, y^0_B + x^\rho_O\, y^0_{/AB}) + \\ &+ T^{\alpha_1,\dots,\alpha_r}_{\beta_1,\dots,\beta_s/\rho B}(x^\rho_A + x^\rho_O\, y^0_A) + T^{\alpha_1,\dots,\alpha_r}_{\beta_1,\dots,\beta_s/A\tau}(x^\tau_B + x^\tau_O\, y^0_B) + T^{\alpha_1,\dots,\alpha_r}_{\beta_1,\dots,\beta_s/AB}. \end{aligned} \tag{11}$$

(3) Without loss of generality we can choose the origin of the abscissa s on W_{y^L} in such a way as to have the left-hand side of equations of Γ vanish.

(4) By analogy see [2].

3. A PROCEDURE FOR THE EXPLICIT CALCULATION OF $T^{\alpha_1,\dots,\alpha_r}_{\beta_1,\dots,\beta_s\,|\,A}$ AND $T^{\alpha_1,\dots,\alpha_r}_{\beta_1,\dots,\beta_s\,|\,AB}$

The quantities lying in (10) and (11) are all known except y^0_A, y^0_B and $y^0_{/AB}$. From (6) and (7), by the substitution of the time parameter;

$$t = t(\tilde{t}, y^L),$$

we obtain:

$$\overset{\circ}{\xi}{}^0 [x^\rho (t(\tilde{t}, y^L), y^L)] = 0 \tag{12}$$

By differerentiating (12) with respect to y^A, we have:

$$\frac{\partial \overset{\circ}{\xi}{}^0}{\partial x^\rho} (x^\rho_A + x^\rho_0 \, y^0_A) = 0 . \tag{13}$$

Since we have calculated the (10) and (11) on W_{y^L} without loss of generality, we obtain from (13) and [3] 4 (21), [3] 4 (30_c):

$$u_\rho (x^\rho_A + x^\rho_0 \, y^0_A) = 0 . \tag{14}$$

Thence we obtain:

$$y^0_A = - \frac{u_\rho \, x^\rho_A}{u_\rho \, x^\rho_0} = \overset{+}{u}_A \frac{Dt}{Ds} \tag{15}$$

under the definition:

$$\overset{+}{u}_A = u_\rho \, x^\rho_A .$$

By further derivation of (13) with respect to y^B we obtain:

$$\begin{aligned} \overset{\circ}{\xi}{}^0_{/AB} &= \overset{\circ}{\xi}{}^0_{/\rho\sigma} (x^\rho_A + x^\rho_0 \, y^0_A)(x^\sigma_B + x^\sigma_0 \, y^0_B) + \\ &+ \frac{\partial \overset{\circ}{\xi}{}^0}{\partial x^\rho} (x^\rho_{/AB} + x^\rho_{/A0} \, y^0_B + x^\rho_{/0B} \, y^0_A + x^\rho_{/00} \, y^0_A \, y^0_B + x^\rho_0 \, y^0_{/AB}) = 0 . \end{aligned} \tag{16}$$

By formulas [3] 4 (21), [3] 4 (30_c), [3] 4 (32) we have on W_{y^L}:

$$\overset{\circ}{\xi}{}^0_{/\rho\sigma} = u_\rho \, A_\sigma + u_\sigma \, A_\rho - \begin{Bmatrix} \lambda \\ \rho\,\sigma \end{Bmatrix} u_\lambda + \begin{Bmatrix} \lambda \\ \sigma\,\rho \end{Bmatrix} u_\lambda = u_\rho \, A_\sigma + u_\sigma \, A_\rho . \tag{17}$$

From (16) and (17), by (15), the orthogonality of u^ρ and A^ρ, and the equality $x^\rho_{AB} = x^\rho_{BA}$ we have the following expression:

$$u_\rho \left[x^\rho_{/AB} + \frac{Dx^\rho_A}{Ds} \overset{+}{u}_B + \frac{Dx^\rho_B}{Ds} \overset{+}{u}_A + x^\rho_{/00} \, \overset{+}{u}_A \, \overset{+}{u}_B \left(\frac{Dt}{Ds} \right)^2 + x^\rho_0 \, y^0_{/AB} \right] = 0$$

which yields:

$$(18)\qquad y^0_{/AB} = u_\rho \left[x^\rho_{/AB} + \frac{Dx^\rho_A}{Ds} u^+_B + \frac{Dx^\rho_B}{Ds} u^+_A + x^\rho_{/OO} u^+_A u^+_B \left(\frac{Dt}{Ds}\right)^2 \right] \frac{Dt}{Ds}.$$

The relations (10), (11), (15) and (18) give us the expressions sought for the optical derivative of the first and second order on W_{y^L}.

4. Comparison between the relativistic gradients of position defined by Lianis and Bressan

Now we want to calculate the expressions of the optical position gradients of the first and the second order. In order to reach this goal, we consider a natural frame, proper and locally geodesic at a point P of W_{y^L}. By the [3] 4 (30_c) and (15) we have:

$$(19)\qquad \overset{\circ}{\xi}{}^\tau_{\natural A} = \frac{\partial \overset{\circ}{\xi}{}^\tau}{\partial x^\rho} (x^\rho_A + x^\rho_O y^0_A) = \overset{\circ}{\Lambda}{}^\tau_\rho (x^\rho_A + u^\rho u^+_A).$$

For the second gradient we have, by [3] 4 (30_c), (15) and (18):

$$(20)\qquad \overset{\circ}{\xi}{}^\tau_{\natural AB} = \frac{\partial^2 \overset{\circ}{\xi}{}^\tau}{\partial x^\rho \partial x^\sigma} (x^\rho_A + u^\rho u^+_A)(x^\sigma_B + u^\sigma u^+_B) +$$

$$+ \overset{\circ}{\Lambda}{}^\tau_\rho \left[x^\rho_{AB} + \frac{Dx^\rho_A}{Ds} u^+_B + x^\rho_{OB} u^+_A \frac{Dt}{Ds} + x^\rho_{OO} u^+_A u^+_B \left(\frac{Dt}{Ds}\right)^2 + \right.$$

$$\left. + u^\rho u_\sigma \left(x^\sigma_{AB} + \frac{Dx^\sigma_A}{Ds} u^+_B + x^\sigma_{OB} u^+_A \frac{Dt}{Ds} + x^\sigma_{OO} u^+_A u^+_B \left(\frac{Dt}{Ds}\right)^2 \right) \right].$$

Expression (20), by [3] 4 (32_a), [3] 4 (32_b) yields [5]

$$(21)\qquad \overset{\circ}{\xi}{}^m_{\natural AB} = -\tfrac{1}{2} \overset{\circ}{\Lambda}{}^m_\rho A_\sigma \overset{1}{g}{}^\rho_\nu (x^\sigma_A x^\nu_B + x^\sigma_B x^\nu_A) +$$

$$+ \overset{\circ}{\Lambda}{}^m_\rho \overset{1}{g}{}^\rho_\sigma \left[x^\sigma_{AB} + \frac{Dx^\sigma_A}{Ds} u^+_B + x^\sigma_{OB} u^+_A \frac{Dt}{Ds} + x^\sigma_{OO} u^+_A u^+_B \left(\frac{Dt}{Ds}\right)^2 \right]$$

$$(22)\qquad \overset{\circ}{\xi}{}^0_{\natural AB} = \overset{\circ}{\Lambda}{}^0_\rho \overset{1}{g}{}^\rho_\sigma \left[x^\sigma_{AB} + \frac{Dx^\sigma_A}{Ds} u^+_B + x^\sigma_{OB} u^+_A \frac{Dt}{Ds} + x^\sigma_{OO} u^+_A u^+_B \left(\frac{Dt}{Ds}\right)^2 \right].$$

Since from [3] 4 (21) we have:

$$\overset{\circ}{\Lambda}{}^0_\rho \overset{1}{g}{}^\rho_\sigma = 0$$

(5) In the formula [3] 4 (32) there is a mistake in a sign that we have corrected.

we can write (21) and (22) as:

$$(23)\qquad \overset{\circ}{\xi}{}^{\tau}_{\S AB} = -\tfrac{1}{2}\overset{\circ}{\Lambda}{}^{\tau}_{\rho} A_\sigma \overset{1}{g}{}^{\rho}_{\nu}(x^\sigma_A x^\nu_B + x^\sigma_B x^\nu_A) + + \overset{\circ}{\Lambda}{}^{\tau}_{\rho}\overset{1}{g}{}^{\rho}_{\sigma}\left[x^\sigma_{AB} + \frac{Dx^\sigma_A}{Ds}u^+_B + x^\sigma_{OB}u^+_A\frac{Dt}{Ds} + x^\sigma_{OO}u^+_A u^+_B\left(\frac{Dt}{Ds}\right)^2\right].$$

If we choose the time-ortogonal parameter t, by the meaning of the vectors $\overset{\circ}{\Lambda}{}^{\tau}_{\rho}$, from the relazions (19), (23), [2] (61), [2] (69), [2] (70) we obtain:

$$(24)\qquad \overset{\circ}{\xi}{}^{\tau}_{\S A} = x^{\tau}_{\| A} = \alpha^{\tau}_{A}$$

$$(25)\qquad \overset{\circ}{\xi}{}^{\tau}_{\S AB} = -\tfrac{1}{2}A_\sigma \overset{1}{g}{}^{\tau}_{\nu}(\alpha^\sigma_A\alpha^\nu_B + \alpha^\sigma_B\alpha^\nu_A) + x^{\tau}_{\| AB} = -\tfrac{1}{2}A_\sigma \overset{1}{g}{}^{\tau}_{\nu}(\alpha^\sigma_A\alpha^\nu_B + \alpha^\sigma_B\alpha^\nu_A) + \alpha^{\tau}_{AB}$$

which give us the relations asked for between the correspondyng relativistic position gradients defined by Lianis and Bressan. We observe that those expressions coincide as far as the first gradient is concerned, while in connection with the second gradient they differ by terms depending on the acceleration. If we put, in according with [4] (19):

$$(26)\qquad \overset{\circ}{C}_{LA} = \overset{\circ}{\xi}_{\tau\S L}\,\overset{\circ}{\xi}{}^{\tau}_{\S A}$$

and

$$(27)\qquad \overset{\circ}{C}_{LAB} = \overset{\circ}{\xi}_{\tau\S L}\,\overset{\circ}{\xi}{}^{\tau}_{\S AB}$$

where $\overset{\circ}{C}_{LA}$ and $\overset{\circ}{C}_{LAB}$ obviously denote the Cauchy-Green tensors with respect to the expressions (19), (23) of the optical gradients, from (24), (25), (26), (27), [4] (19), we obtain:

$$(28)\qquad \overset{\circ}{C}_{LA} = C_{LA}$$

$$(29)\qquad \overset{\circ}{C}_{LAB} = C_{LAB} - \tfrac{1}{2}C_{LA}A_\sigma\alpha^\sigma_B - \tfrac{1}{2}C_{LB}A_\sigma\alpha^\sigma_A .$$

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