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Comparison theorems for a coupled system of singular hyperbolic differential inequalities. II. Time-dependent coefficients with mixed coupled boundary conditions

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Equazioni a derivate parziali. — *Comparison theorems for a coupled system of singular hyperbolic differential inequalities.* II. *Time-dependent coefficients with mixed coupled boundary conditions.* Nota di C. Y. CHAN e EUTIQUIO C. YOUNG, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — L'articolo presenta teoremi di confronto per un sistema accoppiato di disequazioni differenziali iperboliche singolari con coefficienti dipendenti dal tempo e con condizioni al contorno miste e accoppiate. Si danno inoltre i risultati corrispondenti per un sistema accoppiato di disequazioni differenziali ordinarie.

1. INTRODUCTION

The main purpose here is to extend our results in [1] to the case when the uncoupling coefficients of the coupled system of hyperbolic differential inequalities may depend also on the time variable t and when the boundary conditions are mixed and coupled. An an illustration, an example is constructed. At the end of the paper, we give corresponding results for a coupled system of singular ordinary differential inequalities. We refer to [1] for further references.

2. TIME-DEPENDENT COEFFICIENTS WITH MIXED COUPLED BOUNDARY CONDITIONS

Let D be a bounded domain in the real n-dimensional Euclidean space with sufficiently smooth boundary ∂D, $R = D \times (0, T)$ with $T < \infty$, R^- be the closure of R, $x = (x_1, x_2, x_3, \cdots, x_n)$ in D, and $S = \partial D \times (0, T)$. Let us consider the following coupled system

$$(2.1) \qquad u_{tt} + \frac{k}{t} u_t - [a_{ij}(x, t) u_{x_i}]_{x_j} + b(x, t) u - c(x, t) v \geq 0 \quad \text{in} \quad R,$$

$$(2.2) \qquad \partial u / \partial \nu + p(x, t) u - q(x, t) v \geq 0 \quad \text{on} \quad S,$$

$$(2.3) \qquad v_{tt} + \frac{k}{t} v_t - [a_{ij}(x, t) v_{x_i}]_{x_j} + B(x, t) v + C(x, t) u \leq 0 \quad \text{in} \quad R,$$

$$(2.4) \qquad \partial v / \partial \nu + P(x, t) v + Q(x, t) u \leq 0 \quad \text{on} \quad S,$$

where k is a real parameter such that $-\infty < k < \infty$, the repeated indices are to be summed from one to n, and $\partial / \partial \nu = a_{ij} n_i (\partial / \partial x_j)$ is the outward

(*) Nella seduta del 21 aprile 1979.

conormal derivative with $(n_1, n_2, n_3, \cdots, n_n)$ denoting the outward unit normal on S. In the boundary conditions (2.2) and (2.4), we allow

$$-\infty < p(x,t), P(x,t) \leq +\infty,$$

where $p(x_0, t_0) = +\infty$ denotes $u(x_0, t_0) = 0$, and $P(x_1, t_1) = +\infty$ denotes $v(x_1, t_1) = 0$.

We assume that the coefficient matrix (a_{ij}) is symmetric, positive definite and in class $C^1(R^-)$, and the functions b, c, B and C belong to class $C(R^-)$. A solution (u,v) of the coupled system (2.1) and (2.3) belongs to $C^2(R) \cap C^1(R^-)$. As in [1], the following lemma may be established.

LEMMA 1. *If* (u, v) *is a solution of* (2.1) *and* (2.3), *then for any* $k \neq 0$, $u_t(x, 0) = 0 = v_t(x, 0)$.

In this paper, we assume that p, P, q and Q are integrable over S,

$$p(x,t) \leq P(x,t), q(x,t) \geq 0, Q(x,t) \geq 0 \quad \text{on} \quad S,$$

$$b(x,t) \leq B(x,t), c(x,t) \geq 0, C(x,t) \geq 0 \quad \text{in} \quad R,$$

where at least one strict inequality holds somewhere in R. As in [1], we need to distinguish the cases $k \geq 0$ and $k < 0$, and the following condition:

(I) $u > 0$ in R; for $x \in D$, $u(x, T) = 0$, and if $k \leq 0$, then in addition $u(x, 0) = 0$; $v(x_0, t_0) > 0$ for some point (x_0, t_0) in R.

THEOREM 1. *For* $k \geq 0$, *if* (u, v) *is a solution of* (2.1), (2.2), (2.3) *and* (2.4) *such that condition* (I) *holds, then* v *must vanish somewhere in* R.

The proof of the theorem is analogous to that of Theorem 1 of Young [2], and hence is omitted here. To illustrate the above result, let us construct an example.

Example 1. The problem under consideration is given by

$$\begin{aligned}
&u_{tt} - u_{xx} = 0 && \text{for } 0 < x < \pi, 0 < t < \pi,\\
&u(0,t) = 0 = u(\pi,t) && \text{for } 0 < t < \pi,\\
&u(x,0) = 0 = u(x,\pi) && \text{for } 0 < x < \pi,\\
&v_{tt} - v_{xx} + u = 0 && \text{for } 0 < x < \pi, 0 < t < \pi,\\
&v(0,t) = 0 = v(\pi,t) && \text{for } 0 < t < \pi.
\end{aligned}$$

A solution of the above system is

$$(u, v) = (\sin x \ \sin t, (t \sin x \ \cos t)/2).$$

The hypotheses of Theorem 1 are satisfied with $p \equiv +\infty$ for $k = 0$, and hence v must vanish somewhere in the domain $\{(x,t) : 0 < x, t < \pi\}$. Indeed, we readily see that v is zero when $t = \pi/2$.

For $k < 0$, we have the following weaker result. Let $R^* = D \times [0, T)$.

THEOREM 2. *For* $k < 0$, *if* (u, v) *is a solution of* (2.1), (2.2), (2.3) *and* (2.4) *such that condition* (I) *holds, then* v *must vanish somewhere in* R^*.

Proof. Suppose $v > 0$ in R^*. Let

$$w(x, t) = u_t(x, t)\, v(x, t) - u(x, t)\, v_t(x, t).$$

From (2.1) and (2.3),

$$w_t + \frac{k}{t} w \geq [a_{ij}(v u_{x_i} - u v_{x_i})]_{x_j} + (B - b)\, uv + Cu^2 + cv^2 .$$

Integrating this over D and setting

$$I(t) = \int_D w(x, t)\, dx ,$$

we obtain

$$I'(t) + \frac{k}{t} I(t) \geq \int_{\partial D} \left(v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right) ds + \int_D [(B - b)\, uv + Cu^2 + cv^2]\, dx .$$

Let us consider the integrand of the first integral on the right-hand side of the inequality. If $p = +\infty$ at a point on ∂D, then $P = +\infty$ and $u = 0 = v$ there. If $P = +\infty$ at a point on ∂D, then $v = 0$ there and since $v > 0$ in R, it follows that $\partial v / \partial \nu \leq 0$ at that point. If $-\infty < p \leq P < +\infty$, then from the boundary conditions (2.2) and (2.4), the integrand is greater than or equal to

$$(P - p)\, uv + Qu^2 + qv^2 .$$

Thus the first integral is nonnegative, and hence the right-hand side is a positive function of t. By Lemma 1, $I(0) = 0$. Since $u_t(x,T) \leq 0$ and $u(x,T) = 0$, we have $I(T) \leq 0$. As in the proof of Theorem 2 in [1], we conclude that $I(t) < 0$ for $0 < t < T$. Since the integrand of

$$I(t) = \int_D v^2(x, t) \frac{\partial}{\partial t} \left[\frac{u(x, t)}{v(x, t)} \right] dx$$

is continuous, there exists a subdomain D_- of D and a number $\delta > 0$ such that $\frac{\partial}{\partial t}\left(\frac{u}{v}\right) < 0$ in $D_- \times (0, \delta)$, which we denote by R_-. Thus u/v is a decreasing function of t in R_-. Since $u(x, 0) = 0$ in D implies u/v is zero in D at $t = 0$, it follows that u/v is negative in R_-. This contradicts that both u and v are positive in R. Thus the theorem is proved.

As in Theorem 3 of [1], a stronger result may also be given for $k < 0$ if we replace the operator in (2.1) by its adjoint:

$$u_{tt} - \left(\frac{k}{t} u\right)_t - [a_{ij}(x, t)\, u_{x_i}]_{x_j} + b(x, t)\, u - c(x, t)\, v \geq 0 \qquad \text{in } R .$$

3. Ordinary differential inequalities

It is evident that the methods used can be applied to proving analogous comparison theorems for a coupled system of singular ordinary differential inequalities of the form:

$$(3.1)\qquad u'' + \frac{k}{t} u' + b(t)\, u - c(t)\, v \geq 0 \qquad \text{for} \quad 0 < t < T,$$

$$(3.2)\qquad v'' + \frac{k}{t} v' + B(t)\, v + C(t)\, u \leq 0 \qquad \text{for} \quad 0 < t < T,$$

where

$$b(t) \leq B(t) \quad , \quad c(t) \geq 0 \quad , \quad C(t) \geq 0,$$

with at least one strict inequality holding somewhere in the interval $(0, T)$, and the functions b, c, B and C belong to class $C[0, T]$. Such a system of equations with $k = 0$ was studied by Kreith [3].

As in Lemma 1, it can be shown that every solution (u, v) of (3.1) and (3.2) in class $C^2(0, T) \cap C^1[0, T]$ for $k \neq 0$ has the property $u'(0) = 0 = v'(0)$ We can also establish the following two theorems.

THEOREM 3. *If* (u, v) *is a solution of* (3.1) *and* (3.2) *such that*

(*i*) $u > 0$ *for* $0 < t < T$,

(*ii*) $u(T) = 0$, *and if* $k \leq 0$, *then in addition,* $u(0) = 0$,

(*iii*) v *is positive somewhere in the interval* $(0, T)$,

then for $k \geq 0$, v *must vanish somewhere in* $(0, T)$, *and for* $k < 0$, v *must vanish somewhere in* $[0, T)$.

A stronger result for $k < 0$ is given in the following theorem for the adjoint operator of (3.1).

THEOREM 4. *If* (u, v) *is a solution of the coupled system*

$$u'' - \left(\frac{k}{t} u\right)' + b(t)\, u - c(t)\, v \geq 0 \qquad \text{for} \quad 0 < t < T,$$

and (3.2) *for* $k < 0$ *under the hypotheses* (*i*), (*ii*) *and* (*iii*) *of Theorem 3, then* v *must vanish somewhere in* $(0, T)$.

References

[1] C.Y. Chan and E.C. Young (1979) – *Comparison theorems for a coupled system of singular hyperbolic differential inequalities.* I. *Time-independent uncoupling coefficients*, « Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Natur. » *66*, 250–254.

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[3] K. Kreith (1975) – *Rotation properties of a class of second order differential systems*, « J. Differential Equations », *17*, 395-405.