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On oscillatory solutions of a forced second order nonlinear differential equation

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Equazioni differenziali ordinarie. — *On oscillatory solutions of a forced second order nonlinear differential equation.* Nota di N. PARHI, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si dimostra un teorema sul comportamento asintotico delle soluzioni di un'equazione non lineare del secondo ordine.

In this paper we are concerned with the behaviour of solutions of the differential equation

$$y'' + p(t) g(y) = f(t), \tag{1}$$

where p and f are real-valued continuous functions on $[0, \infty)$ with $p(t) \geq 0$ for $t \in [0, \infty)$ and g is a real-valued continuous function on $(-\infty, \infty)$. In a recent paper [1], Hammett considered

$$(r(t) y')' + p(t) g(y) = f(t), \tag{2}$$

where r was a real-valued continuous function on $[0, \infty)$ with $r(t) > 0$ and p, g and f were as above. He proved, under certain conditions on r, p, g and f, that a non-oscillatory solution of (2) goes to zero as $t \to \infty$. In this note we prove a theorem for (1), a particular case of (2), and give examples to illustrate that our theorem can be applied to a class of differential equations to which Hammett's theorem cannot be applied.

A solution $y(t)$ of (1) existing on $[\tau, \infty), \tau \geq 0$, is said to be non-oscillatory if there exists a $t_1 \geq \tau$ such that $y(t) \neq 0$ for $t > t_1$ and is said to be oscillatory if it is not nonoscillatory. In the following we prove the theorem.

THEOREM. *Consider Eqn.* (1). *Let the following conditions be satisfied*:

(*i*) $\int_0^\infty p(t)\, dt = \infty$,

(*ii*) $y g(y) > 0$ *for* $y \neq 0$,

(*iii*) $g'(y)$ *exists and* ≥ 0 *for all* $y \neq 0$,

(*iv*) $\int_0^\infty |f(t)|\, dt < \infty$.

If $y(t)$ *is a solution of* (1), *existing on* $[\tau, \infty)$, *then* $\lim_{t \to \infty} y(t) = 0$ *or* $y(t)$ *is oscillatory.*

(*) Nella seduta del 12 maggio 1979.

Proof. Let $y(t)$ be nonoscillatory. So it is ultimately positive or ultimately negative. Let $y(t)$ be ultimately positive. The case when $y(t)$ is ultimately negative can be treated similarly. Let $y(t) > 0$ for $t \geq t_0 > \tau$. To prove that $\lim_{t\to\infty} y(t) = 0$, it is enough to show that $\limsup_{t\to\infty} y(t) = 0$. If possible, let $\limsup_{t\to\infty} y(t) = k$, $k > 0$. We consider three cases, viz, (*i*) $y'(t)$ is ultimately positive, (*ii*) $y'(t)$ is oscillatory and (*iii*) $y'(t)$ is ultimately negative; and derive a contradiction in each of these cases.

Let $y'(t) > 0$ for $t \geq t_1 > t_0$. So $y(t)$ is increasing for $t \geq t_1$. Integrating

$$y''(t)g^{-1}(y(t)) + p(t) = f(t)g^{-1}(y(t)) \tag{3}$$

from t_1 to t we get

$$[y'(s)g^{-1}(y(s))]_{t_1}^{t} + \int_{t_1}^{t} g^{-2}(y(s))g'(y(s))(y'(s))^2\,ds \, . \tag{4}$$

$$+ \int_{t_1}^{t} p(s)\,ds = \int_{t_1}^{t} f(s)g^{-1}(y(s))\,ds \, .$$

So

$$\int_{t_1}^{t} p(s)\,ds \leq \int_{t_1}^{t} f(s)g^{-1}(y(s))\,ds + y'(t_1)g^{-1}(y(t_1))$$

$$\leq \int_{t_1}^{t} |f(s)|g^{-1}(y(s))\,ds + y'(t_1)g^{-1}(y(t_1))$$

$$\leq g^{-1}(y(t_1))\left[\int_{t_1}^{t} |f(s)|\,ds + y'(t_1)\right].$$

Since this is true for every $t \geq t_1$, from assumption (*iv*) it follows that $\int_{t_1}^{\infty} p(t)\,dt < \infty$. This contradicts assumption (*i*).

Let $y'(t)$ be oscillatory and let $\langle t_n \rangle$ be a sequence of zeros of $y'(t)$ such that $t_0 < t_1 < t_2 < \cdots$ and $t_n \to \infty$ as $n \to \infty$. Let $\sigma > t_0$. Integrating (3) from σ to t_n, we get

$$\int_{\sigma}^{t_n} p(t)\,dt \leq \int_{\sigma}^{t_n} f(t)g^{-1}(y(t))\,dt + y'(\sigma)g^{-1}(y(\sigma))$$

$$\leq \int_{\sigma}^{t_n} |f(t)|g^{-1}(y(t))\,dt + |y'(\sigma)|g^{-1}(y(\sigma)) \, .$$

From the first mean-value theorem of integral calculus (cf. [2], pp. 617) it follows that

$$(5) \qquad \int_{\sigma}^{t_n} p(t)\,dt \le g^{-1}(y(\xi)) \int_{\sigma}^{t_n} |f(t)|\,dt + |y'(\sigma)|\, g^{-1}(y(\sigma))$$

where $\sigma \le \xi \le t_n$. Clearly, $\sigma \le \limsup_{t_n \to \infty} \xi \le \infty$. Since $\limsup_{t \to \infty} y(t) = k$, $\limsup_{t_n \to \infty} g^{-1}(y(\xi)) < \infty$. Now taking lim sup, as $t_n \to \infty$, in (5), we get $\int_{\sigma}^{\infty} p(t)\,dt < \infty$. This again contradicts assumption (*i*).

Finally, let $y'(t)$ be ultimately negative. Let $y'(t) < 0$ for $t \ge t_1 > t_0$. So $y(t)$ is decreasing for $t \ge t_1$. From (4) we get,

$$(6) \qquad \begin{aligned} \int_{t_1}^{t} p(s)\,ds &\le \int_{t_1}^{t} f(s)\, g^{-1}(y(s))\,ds - y'(t)\, g^{-1}(y(t)) \\ &\le \int_{t_1}^{t} |f(s)|\, g^{-1}(y(s))\,ds - y'(t)\, g^{-1}(y(t)) \\ &\le g^{-1}(y(t)) \left[\int_{t_1}^{t} |f(s)|\,ds - y'(t) \right]. \end{aligned}$$

We claim that $-\infty < \liminf_{t \to \infty} y'(t)$. If not, $\liminf_{t \to \infty} y'(t) = -\infty$. Choose $K > \int_{t_1}^{\infty} |f(t)|\,dt$. Since $\liminf_{t \to \infty} y'(t) = -\infty$, there exists a $t_2 > t_1$ such that $y'(t_2) < -K$. Now integrating (1) from t_2 to t we get

$$y'(t) \le y'(t_2) + \int_{t_2}^{t} |f(s)|\,ds\,.$$

Further integration from t_2 to t yields

$$y(t) \le y(t_2) + (t - t_2)\, y'(t_2) + t \int_{t_2}^{t} |f(s)|\,ds\,,$$

that is,

$$\frac{y(t)}{t} \le \frac{y(t_2)}{t} + \left(1 - \frac{t_2}{t}\right) y'(t_2) + \int_{t_1}^{t} |f(s)|\,ds\,.$$

Hence

$$\limsup_{t\to\infty} \frac{y(t)}{t} \leq y'(t_2) + \int_{t_1}^{\infty} |f(s)|\, ds$$
$$< -K + \int_{t_1}^{\infty} |f(s)|\, ds$$
$$< 0 .$$

Consequently, $y(t) < 0$ for sufficiently large t. This contradicts the fact that $y(t) > 0$ for $t > t_0$. This proves our assertion. Now taking lim sup, as $t \to \infty$, in (6), we get $\int_{t_1}^{\infty} p(s)\, ds < \infty$ a contradiction.

This completes the proof of the theorem.

Following examples illustrate above theorem. Hammett's theorem cannot be applied to any of these examples.

Esamples

(*a*) $y'' + \frac{1}{t} y = \frac{2}{t^3} + \frac{1}{t^2}, \qquad t \geq 1 .$

(*b*) $y'' + \frac{1}{t} y^3 = \frac{2}{t^3} + \frac{1}{t^4}, \qquad t \geq 1 .$

In each of the above examples, $y(t) = 1/t$ is a nonoscillatory solution going to zero as $t \to \infty$.

REFERENCES

[1] M. E. HAMMETT (1971) – *Nonoscillation properties of a nonlinear differential equation*, « Proc. Amer. Math. Soc. », *30*, 92–96.

[2] E. W. HOBSON (1957) – *The theory of functions of a real variable and the theory of Fourier Series*, Vol. I, Dover.