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On oscillatory property of bounded solutions of functional differential equations

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Equazioni differenziali ordinarie. — *On oscillatory property of bounded solutions of functional differential equations* (*). Nota di LU-SAN CHEN, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — L'A. considera l'equazione

$$[r(t)\,x^{(n-m)}(t)]^{(m)} + a(t)\,h(x(t))\,p(x'(t)) + b(t)\,f(x[g_1(t)],\cdots,x[g_l(t)]) = c(t)$$

e studia il comportamento oscillatorio delle soluzioni limitate.

1. INTRODUCTION

Recently, Staikos-Sficas [5] discussed the oscillatory behavior of bounded solutions of the following differential equation

$$[r(t)\,x^{(n-m)}(t)]^{(m)} + b(t)\,f(x[g_1(t)],\cdots,x[g_l(t)]) = c(t)$$

where $t \in [t_0, \infty)$, l, m, n are positive integers such that $n \geq 2$, $1 \leq m \leq n-1$, $b(t)$, $c(t)$, $g_i(t)$, $(i = 1, 2, \cdots, l)$ and $r(t)$ are real-valued continuous functions on $[t_0, \infty)$ and $f(y_1, \cdots, y_l)$ is a real-valued continuous function on R^l and also assume that $r(t) > 0$, such that $\int^{\infty} \frac{dt}{r(t)} = \infty$, $y_i f(y_1, \cdots, y_l) > 0$ for $y_i \neq 0$, $(i = 1, 2, \cdots, l)$ and $\lim_{t \to \infty} g_i(t) = \infty$, $(i = 1, 2, \cdots, l)$.

In this paper, we extend Staikos-Sficas's result to the following more general functional differential equations:

$$(1) \qquad [r(t)\,x^{(n-m)}(t)]^{(m)} + a(t)\,h(x(t))\,p(x'(t)) + b(t)\,f(x[g_1(t)],\cdots,x[g_l(t)]) = c(t)$$

where $t \in [t_0, \infty)$, l, m, n are positive integers such that $n \geq 2$, $1 \leq m \leq n-1$.

In what follows we restrict our consideration to those solutions of (1) which exist on some half-line $[t_x, \infty)$. Also we shall consider the oscillatory character of the solutions in the usual sence, i.e. a solution is called *oscillatory* if it has arbitratily large zeros. Otherwise it is called *nonoscillatory*.

In addition, the following assumptions will be made for the rest of this paper.

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Assumptions:

(i) *$a(t)$, $b(t)$ and $c(t)$ are continuous real-valued functions on* $[t_0, \infty)$, $a(t) \geq 0$, $b^+(t) = \max\{b(t), 0\}$ *and* $b^-(t) = \text{mas}\{-b(t), 0\}$;

(ii) *$r(t)$ is a continuous real-valued function on* $[t_0, \infty)$ *such that* $\int^{\infty} \frac{dt}{r(t)} = \infty$ *and* $r(t) > 0$;

(iii) *$g_i(t)$, $(i = 1, 2, \cdots, l)$ are continuous real-valued functions on* $[t_0, \infty)$ *such that* $\lim_{t\to\infty} g(t) = \infty$;

(iv) *$h(t)$ is a continuous real-valued function on* $(-\infty, \infty)$ *such that* $xh(x) > 0$ *for* $x \neq 0$;

(v) *$p(x')$ is a continuous real-valued function on* $(-\infty, \infty)$ *such that* $k_2 \geq p(x') \geq k_1 > 0$;

(vi) *$f(y_1, \cdots, y_l)$ is a continuous real-valued function on* R^l *such that* $y_i f(y_1, \cdots, y_l) > 0$ *for* $y_i \neq 0$, $(i = 1, 2, \cdots, l)$.

In order to obtain our result it is convenient to give the following lemma.

LEMMA (Staikos-Sficas [3], [4]). *Let $u(t)$ be the solution of the linear differential equation*

$$u' - \frac{\alpha}{t} u + \frac{H(t)}{t} = 0 \qquad on \quad [T, \infty), T > 0,$$

satisfying $u(T) = 0$, *where α is a positive constant and* $H(t)$ *is continuous on* $[T, \infty)$. *If* $\lim_{t\to\infty} |H(t)| = H^*$ *exists in the extended real line* R^*, *then* $\lim_{t\to\infty} |u(t)| = u^*$ *exists in* R^*. *In particular* $H^* = \infty$ *implies* $u^* = \infty$.

2. MAIN RESULT

THEOREM. *Let equation* (1) *satisfy* (i)–(vi) *and, in addition, any one of the following holds*:

For some integer k, $0 \leq k \leq m-1$ and for every constant $\mu_1 > 0$, $\mu_2 > 0$ *and* $\mu_3 > 0$

$$\int^{\infty} t^k [\mu_3 a(t) + \mu_1 b^+(t) - b^-(t) - \mu_2 c(t)] dt = \infty,$$

or

$$\int^{\infty} t^k [-\mu_3 a(t) + \mu_1 b^-(t) - b^+(t) + \mu_2 c(t)] dt = \infty.$$

Then every bounded solution of $x(t)$ of (1) *is either oscillatory or such that*

$$\liminf_{t\to\infty} |x(t)| = 0.$$

Proof. We suppose, now that the conclusion of the theorem is not valid, when there exists a bounded nonoscillatory solution $x(t)$ of (1) with $\liminf_{t\to\infty} |x(t)| \neq 0$. Without any loss of generality, that $x(t) > 0$ eventually. The case in which $x(t) < 0$ can be treated similarly.

From (iii), (iv) and (vi), there exist a $T > \max\{t_0, 0\}$, and positive constants c_1, c_2, h_1 and h_2 such that

$$h_1 \leq h(x(t)) \leq h_2, \tag{2}$$

$$c_1 \leq f(x[g_1(t)], \cdots, x[g_l(t)]) \leq c_2, \tag{3}$$

for $t \geq T$.

If

$$u_{ij}(t) = \int_T^t s^i [r(s)\, x^{(n-m)}(s)]^{(j)}\, ds, \qquad 1 \leq i \leq j.$$

An integration by pasts yields

$$\begin{aligned} u_{ij}(t) &= t^i [r(t)\, x^{(n-m)}(t)]^{(j-1)} \\ &\quad - T^i [r(s)\, x^{(n-m)}(s)]^{(j-1)}_{s=T} - i \int_T^t s^{i-1} [r(s)\, x^{(n-m)}(s)]^{(i-1)}\, ds \\ &= t u'_{i-1,j-1}(t) - i u_{i-1,j-1}(t) - T^i [r(s)\, x^{(n-m)}(s)]^{(j-1)}_{s=T}, \end{aligned}$$

which shows that $u_{i-1,j-1}(t)$ is a solution of the differential equation

$$u' - \frac{i}{t} u + \frac{H_{ij}(t)}{t} = 0 \tag{4}$$

where $H_{ij}(t) = -u_{ij}(t) - T^i [r(s)\, x^{(n-m)}(s)]^{(j-1)}_{s=T}$. Obviously, $u_{i-1,j-1}(t)$ satisfies the initial condition $u_{i-1,j-1}(T) = 0$. Since

$$\begin{aligned} H_{k,m}(t) &= -u_{k,m}(t) - T^k [r(s)\, x^{(n-m)}(s)]^{(m-1)}_{s=T} \\ &= \int_T^s s^k \{a(s)\, h(x(s))\, p(x'(s)) + b(s) f(x[g_1(s)], \cdots, x[g_l(s)]) \\ &\quad - c(s)\}\, ds - T^k [r(s)\, x^{(n-m)}(s)]^{(m-1)}_{s=T} \end{aligned}$$

consequently, from (V), (2) and (3), we have

$$\begin{aligned} H_{k,m}(t) \geq c_2 \int_T^t s^k \left[\frac{h_1 k_1}{c_2} a(s) + \frac{c_1}{c_2} b^+(s) - b^-(s) - \frac{1}{c_2} c(s) \right] ds \\ - T^k [r(s)\, x^{(n-m)}(s)]^{(m-1)}_{s=T} \end{aligned}$$

and

$$H_{k,m}(t) \leq c_2 \int_T^t s^k \left[\frac{h_2 k_2}{c_2} a(s) + b^+(s) - \frac{c_1}{c_2} b^-(s) - \frac{1}{c_2} c(s) \right] ds$$

$$- T^k [r(s) x^{(n-m)}(s)]_{s=T}^{(m-1)}$$

for every $t \geq T$.

Hence, by assumptions of the theorem,

$$\lim_{t \to \infty} | H_{k,m}(t) | = \infty$$

and consequently

$$\lim_{t \to \infty} | u_{k-1,m-1}(t) | = \infty .$$

Since

$$H_{k-1,m-1}(t) = - u_{k-1,m-1}(t) - T^{k-1} [r(s) x^{(n-m)}(s)]_{s=T}^{(m-2)} ,$$

whe have

$$\lim_{t \to \infty} | H_{k-1,m-1}(t) | = \infty .$$

Applying, now the lemma again, we obtain

$$\lim_{t \to \infty} | u_{k-2,m-2}(t) | = \infty .$$

Following the same procedure, we obtain finally

$$\lim_{t \to \infty} | u_{0,m-k}(t) | = \infty ,$$

which gives that

$$\lim_{t \to \infty} | r(t) x^{(n-m)}(t) | = \infty .$$

Case 1.

$$\lim_{t \to \infty} r(t) x^{(n-m)}(t) = \infty .$$

If so, there exists $T_1 \geq T$ such that

$$r(t) x^{(n-m)}(t) > 1 \qquad \text{for} \quad t \geq T_1 . \tag{5}$$

Dividing (5) by $r(t)$, integrating from T_1 to t, we have

$$x^{(n-m-1)}(t) > x^{(n-m-1)}(T_1) + \int_{T_1}^t \frac{ds}{r(s)} \qquad \text{for} \quad t \geq T_1 ,$$

which by (ii) gives

$$\lim_{t \to \infty} x^{(n-m-1)}(t) = \infty$$

and consequently

$$\lim_{t \to \infty} x(t) = \infty ,$$

which is a contradiction.

Case 2.

$$\lim_{t\to\infty} r(t)\, x^{(n-m)}(t) = -\infty .$$

In this case, following the same procedure we obtain the contradiction $\lim_{t\to\infty} x(t) = -\infty$.

REMARK 1. For the case $a(t) \equiv 0$, Theorem becomes a result of Staikos-Sficas [5].

We conclude with the statement that this theorem can be extended to the following equations

$$[r(t)\, x^{(n-1)}(t)]' + a(t)\, h(x(t))\, p(x'(t)) + b(t) f(x[g(t)]) = c(t) \tag{6}$$

and

$$[r(t)\, x'(t)]^{(n-1)} + a(t)\, h(x(t))\, p(x'(t)) + b(t) f(x[g(t)]) = c(t) \tag{7}$$

which are the special cases of (1) with $m = 1$, $l = 1$; and $n = n - 1$, $l = 1$ respectively, and the above theorem leads to a generalization of a result of the author [1].

REMARK 2. Kusano-Onose [2] considered the equations

$$[r(t)\, x^{(n-1)}(t)]' + b(t) f(x[g(t)]) = c(t)$$

and

$$[r(t)\, x'(t)]^{(n-1)} + b(t) f(x[g(t)]) = c(t)$$

which are the special cases of (6) and (7) with $a(t) \equiv 0$, respectively.

Also, we may extend to the following equation

$$[r(t)\, x'(t)]' + a(t)\, h(x(t))\, p(x'(t)) + b(t) f(x[g(t)]) = c(t)$$

which is a special case of both (6) and (7) with $n = 2$.

REMARK 3. It is easy to extend our result to the following differential equation

$$[r(t)\, x^{(n-m)}(t)]^{(m)} + a(t)\, h(x(t))\, p(x'(t)) + F(t; x\langle\sigma_0(t)\rangle, \cdots, x^{(n-1)}\langle\sigma_{n-1}(t)\rangle) = 0$$

where $x^{(i)}\langle\sigma_i(t)\rangle \equiv (x^{(i)}[\sigma_{i1}(t)], \cdots, x^{(i)}[\sigma_{im_i}(t)])$ and

$$\lim_{t\to\infty} \sigma_{ij}(t) = \infty, \qquad i = 0, 1, \cdots, n-1; \qquad j = 1, 2, \cdots, m_i .$$

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