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Periodic solutions of a certain fourth order differential equation

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Equazioni differenziali ordinarie. — *Periodic solutions of a certain fourth order differential equation.* Nota di JAMES O. C. EZEILO e HAROON O. TEJUMOLA, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Gli Autori dànno un teorema di esistenza di almeno una soluzione periodica per un'equazione differenziale non lineare del quarto ordine.

1. INTRODUCTION

Consider the fourth order constant—coefficient differential equation

$$x^{(4)} + a_1 \dddot{x} + a_2 \ddot{x} + a_3 \dot{x} + a_4 x = 0 . \tag{1.1}$$

Its corresponding auxiliary equation

$$r^4 + a_1 r^3 + a_2 r^2 + a_3 r + a_4 = 0 \tag{1.2}$$

has no purely imaginary roots $r = i\lambda \, (\lambda \neq 0)$ if

$$\chi_1 \equiv \lambda^4 - a_2 \lambda^2 + a_4 \neq 0 , \tag{1.3}$$

or if

$$\chi_2 \equiv a_1 \lambda^2 - a_3 \neq 0 . \tag{1.4}$$

Note that χ_1 may be rearranged in the form

$$\chi_1 = (\lambda^2 - \tfrac{1}{2} a_2)^2 + a_4 - \tfrac{1}{4} \alpha_2^2$$

so that (1.3) holds if, for example,

$$a_4 > \tfrac{1}{4} a_2^2 . \tag{1.5}$$

It is also easy to see that (1.4) will hold if

$$a_1 a_3 < 0 , \qquad a_4 \neq 0 \tag{1.6}$$

the condition on a_4 here being necessary in order to ensure that $\lambda \neq 0$. Now we know from the general theory that if (1.2) has no purely imaginary roots then the equation

$$x^{(4)} + a_1 \dddot{x} + a_2 \ddot{x} + a_3 \dot{x} + a_4 x = p(t) \tag{1.7}$$

(*) Nella seduta del 12 maggio 1979.

with $a_1, \cdots, a_4$ as before but with p dependent on and ω-periodic in t (for some $\omega > 0$) necessarily has an ω-periodic solution; and therefore (1.7) has an ω-periodic solution subject to (1.5) for arbitrary a_1 and a_3 or subject to (1.6) for arbitrary a_2. There are now in the literature some extensions of (1.6) in one form or other of the existence result for equations (1.7) in which $a_1, \cdots, a_4$ are not necessarily constants (see for example, [1, Theorem 1], [2]), but no similar extension exists, as far as we are aware, for (1.7) subject to (1.5).

The present paper is concerned solely with this outstanding case (1.5), but in the context of the equation

$$(1.8) \qquad x^{(4)} + a_1 \dddot{x} + f(x, \dot{x}, \ddot{x}, \dddot{x})\, \ddot{x} + g(x)\, \dot{x} + h(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x})$$

where a_1 is a constant as before but f, g, h and p are continuous in their respective arguments, with p ω-periodic in t (that is $p(t + \omega, x, y, z, u) = p(t, x, y, z, u)$ for arbitrary t, x, y, z and u). Our object is to establish the following result:

THEOREM. *Suppose that*

(i) *there exists a constant* $a_2 \geq 0$ *such that*

$$(1.9) \qquad |f(x, y, z, u)| \leq a_2 \qquad \textit{for all} \quad x, y, z, u,$$

$$(1.10) \qquad \beta \equiv \inf_{|x| \geq 1} x^{-1} h(x) > \tfrac{1}{4} a_2^2$$

(ii) *there are constants* $A_0 \geq 0, A_1 \geq 0$ *such that*

$$(1.11) \qquad |p(t, x, y, z, u)| \leq A_0 + A_1 (|x| + |y| + |z|)$$

for all t, x, y, z *and* u.

Then there exists a constant $\varepsilon_0 > 0$ *such that, if* $A_1 \leq \varepsilon_0$ *then* (1.8) *admits of at least one* ω*-periodic solution.*

Note that there are no conditions on a_1 and g.

2. SOME PRELIMINARIES.

As in [1] and [2] the proof of the theorem rests on the Leray-Schauder technique, a convenient starting point of which shall be the parameter (μ)-dependent equation:

$$(2.1) \qquad x^{(4)} + a_1 \dddot{x} + \{(1 - \mu)\, a_2 + \mu f(x, \dot{x}, \ddot{x}, \dddot{x})\}\, \ddot{x} + \mu g(x)\, \dot{x} + \\ + (1 - \mu)\, \beta x + \mu h(x) = \mu p(t, x, \dot{x}, \ddot{x}, \dddot{x}), \qquad (0 \leq \mu \leq 1).$$

Note that when $\mu = 1$ (2.1) reduces to the original equation (1.1). Also, when $\mu = 0$ it reduces to the linear equation

$$x^{(4)} + a_1 \dddot{x} + a_2 \ddot{x} + \beta x = 0$$

which, in view of the condition: $\beta > \frac{1}{4} a_2^2$ (from (1.10), has no non-trivial ω-periodic solution. The equation (2.1) thus has the base features expected of parameter—dependent equations for the application of the usual Leray-Schauder fixed point technique, and hence, in order to establish the theorem, it remains only for us to show that there is a constant $D > 0$ independent of μ such that

$$|x(t)| \leq D, |\dot{x}(t)| \leq D, |\ddot{x}(t)| \leq D, |\dddot{x}(t)| \leq D \qquad (0 \leq t \leq \omega) \tag{2.2}$$

for any ω-periodic solution $x(t)$ of (2.1).

Before proceeding to the actual verification of (2.2) we shall introduce some notations. Throughout what follows the $D, D_0, D_1, \cdots$ whenever they occur are positive constants whose magnitude depend only on the constants $a_1, a_2, \beta, \omega, A_0$ as well as on the functions f, g and h, but not on μ. The numbered D's: $D_0, D_1, D_2, \cdots$ retain the same identity throughout while the unnumbered D's are not necessarily the same each time they occur.

3. Verification of the bounds for $|x(t)|$ and $|\dot{x}(t)|$ in (2.2).

We shall take (2.1) in the more compact form:

$$x^{(4)} + a_1 \dddot{x} + f_\mu(x, \dot{x}, \ddot{x}, \dddot{x}) \ddot{x} + \mu g(x) \dot{x} + h_\mu(x) = \mu p(t, x, \dot{x}, \ddot{x}, \dddot{x}), \tag{3.1}$$

$$(0 \leq \mu \leq 1)$$

by setting

$$f_\mu = (1 - \mu) a_2 + \mu f(x, \dot{x}, \ddot{x}, \dddot{x})$$

$$h_\mu = (1 - \mu) \beta x + \mu h(x)$$

Note here that

$$|f_\mu(x, \dot{x}, \ddot{x}, \dddot{x})| \leq a_2 \qquad \text{always}, \tag{3.2}$$

by (1.9). Also

$$x^{-1} h_\mu(x) \geq \beta \qquad \text{for} \quad |x| \geq 1$$

by (1.10), which in turn can be shown to lead to:

$$x h_\mu(x) \geq \beta x^2 - D_0 \qquad \text{for all} \quad x. \tag{3.3}$$

Throughout what follows in this paper let $x = x(t)$ be an arbitrary ω-periodic solution of (3.1) and $V = V(t)$ the function correspondingly defined by

$$V = \dot{x}(\ddot{x} + \tfrac{1}{2} a_1 \dot{x}) - x(\dddot{x} + a_1 \ddot{x}) - \mu \int_0^x s g(s) \, ds.$$

An elementary differentiation gives that

$$\dot{V} = U_1 - \mu x p \tag{3.4}$$

where

$$(3.5)\qquad U_1 \equiv \ddot{x}^2 + x\ddot{x}f_\mu + x\,h_\mu(x)\,.$$
$$\geq \ddot{x}^2 - a_2\,|x|\;|\ddot{x}| + \beta x^2 - D_0$$

by (3.2) and (3.3). Subject to the condition (1.10) it is possible to obtain the following more refined estimate for U_1:

$$(3.6)\qquad U_1 \geq D_1(\ddot{x}^2 + x^2) - D_0$$

for some suitably fixed D_1. Indeed by (3.5),

$$\begin{aligned} U_1 - \{D_1(\ddot{x}^2 + x^2) - D_0\} &\geq (1 - D_1)\,\ddot{x}^2 - a_2\,|x|\;|\ddot{x}| + (\beta - D_1)\,x^2 \\ &= (1 - D_1)\,\{|\ddot{x}| - \tfrac{1}{2}a_2(1 - D_1)^{-1}\,|x|\}^2 + \tfrac{1}{4}(1 - D_1)^{-1} \\ &\quad [(4\beta - a_2^2) - 4\,D_1(1 + \beta) + 4\,D_1^2]\,x^2 \geq 0\,, \end{aligned}$$

in D_1 is fixed such that

$$(3.7)\qquad D_1 < \min\{1\,,\,\tfrac{1}{4}(4\beta - a_2^2)(1 + \beta)^{-1}\}\,.$$

The term $(4\beta - a_2^2)$ here is positive by (1.10), so that the choice of a positive D_1 satisfying (3.7) is possible. We can therefore assume (3.6) subject to (3.7) on D_1. Hence, by (3.4) and (3.6)

$$\dot{V} \geq D_1(\ddot{x}^2 + x^2) - |x|\,|p| - D_0 \qquad \text{always;}$$

from which, on integration, we obtain ($V(t)$ being ω-periodic) that

$$(3.8)\qquad 0 \geq D_1\int_0^\omega (\ddot{x}^2 + x^2)\,dt - \int_0^\omega |x|\,|p|\,dt - D_0\,\omega\,.$$

But, by (1.11),

$$|x|\,|p| \leq A_0\,|x| + \tfrac{1}{2}A_1(\ddot{x}^2 + \ddot{x}^2 + 4\,x^2)$$

Also

$$(3.9)\qquad \int_0^\omega \dot{x}^2\,dt \leq \tfrac{1}{4}\,\omega^2\,\pi^{-2}\int_0^\omega \ddot{x}^2\,dt$$

since x is ω-periodic. Thus (3.8) implies at once that

$$(3.10)\qquad \int_0^\omega \{D_1 - \tfrac{1}{2}A_1(1 + \omega^2\,\pi^2)\}\,\ddot{x}^2\,dt + \int_0^\omega (D_1 - 2\,A_1)\,x^2\,dt \leq$$
$$\leq \int_0^\omega A_0\,|x|\,dt + D_0\,\omega\,.$$

Now assume A_1 fixed such that

$$D_1 - A_1 (1 + \omega^2 \pi^{-2}) \geq 0 \quad , \quad D_1 - 4 A_1 \geq 0 . \tag{3.11}$$

Then, by (3.10),

$$D_1 \int_0^\omega (\ddot{x}^2 + x^2) \, dt \leq 2 A_0 \int_0^\omega |x| \, dt + 2 D_0 \omega$$

from which it follows that

$$\int_0^\omega x^2 \, dt \leq D_2 \quad , \quad \int_0^\omega \ddot{x}^2 \, dt \leq D_3 \tag{3.12}$$

for some D_2 , D_3. We also have, in view of (3.9) that

$$\int_0^\omega \dot{x}^2 \, dt \leq D_4 \tag{3.13}$$

for some D_4.

It is easy to see from (3.12) that $|x(\tau_0)| \leq (\omega^{-1} D_2)^{1/2}$ for some $\tau_0 \in [0 , \omega]$; for otherwise, that is if $x^2(t) > \omega^{-1} D_2$ for all $t \in [0 , \omega]$, then $\int_0^\omega x^2 \, dt > D_2$ which would contradict the first inequality in (3.12). We thus have from the identity

$$x(t) = x(\tau_0) + \int_0^\omega \dot{x}(s) \, ds$$

that

$$\max_{0 \leq t \leq \omega} |x(t)| \leq (\omega^{-1} D_2)^{1/2} + \int_{\tau_0}^{\tau_0 + \omega} |\dot{x}(s)| \, ds$$

$$\leq D + \omega^{1/2} \left(\int_{\tau_0}^{\tau_0 + \omega} \dot{x}^2(s) \, ds \right)^{1/2}$$

by Schwarz's inequality. It follows then from (3.13) that

$$\max_{0 \leq t \leq \omega} |x(t)| \leq D . \tag{3.14}$$

Analogously, the fact that $\dot{x}(\tau_1) = 0$ for some $\tau_1 \in [0 , \omega]$ combined with the fact that $\int_0^\omega \ddot{x}^2 \, dt \leq D_3$ (in 3.12) yields the result

$$\max_{0 \leq t \leq \omega} |\dot{x}(t)| \leq D . \tag{3.15}$$

4. COMPLETION OF PROOF: BOUNDS FOR $|\ddot{x}(t)|$ AND $|\dddot{x}(t)|$.

To estimate a bound for $|\dddot{x}(t)|$ reset (3.1) in the form

$$x^{(4)} + \alpha_1 \dddot{x} = \Psi \tag{4.1}$$

where $\Psi = \mu p - h_\mu(x) - \mu g(x)\dot{x} - \ddot{x} f_\mu$, and note that, by (1.11), (3.2), (3.14) and (3.15),

$$|\Psi| \leq D_5 + D_6 |\ddot{x}|, \tag{4.2}$$

since A_1 is now fixed as a D by (3.11). Now multiply both sides of (4.1) by $x^{(4)}$ and integrate from $t = 0$ to $t = \omega$. We will then have, in view of the (assumed) ω-periodicity of x and the result (4.2), that

$$\int_0^\omega \{x^4(t)\}^2 \, dt \leq D_5 \int_0^\omega |x^{(4)}(t)| \, dt + D_6 \int_0^\omega |\ddot{x}(t)| \, |x^{(4)}(t)| \, dt$$

from which, on applying the Schwarz's inequality to the terms on the right hand side and using the second result in (3.12) as applicable, we obtain that

$$\int_0^\omega \{x^{(4)}(t)\}^2 \, dt \leq D_7 \left[\int_0^\omega \{x^{(4)}(t)\}^2 \, dt \right]^{1/2}.$$

Hence

$$\int_0^\omega \{x^{(4)}(t)\}^2 \, dt \leq D_8 = D_7^2 \tag{4.3}$$

which in turn, since

$$\int_0^\omega \dddot{x}^2 \, dt \leq \tfrac{1}{4} \omega^2 \pi^{-2} \int_0^\omega \{x^{(4)}\}^2 \, dt,$$

also gives that

$$\int_0^\omega \dddot{x}^2(t) \, dt \leq D_9. \tag{4.4}$$

The fact that $\ddot{x}(\tau) = 0$ at some $\tau \in [0, \omega]$, $x(t)$ being ω-periodic, combines with (4.4) in the usual manner to give that

$$\max_{0 \leq t \leq \omega} |\ddot{x}(t)| \leq D.$$

Likewise the fact that $\dddot{x}(\tau) = 0$ at some $\tau \in [0, \omega]$ combines with (4.3) to yield the remaining estimate:

$$\max_{0 \leq t \leq \omega} |\dddot{x}(t)| \leq D.$$

The verification of (2.2) is thus completed subject to the restriction (3.11) on A_1. The theorem then follows.

References

[1] J. O. C. Ezeilo (1977) – « Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. e nat. », ser VIII, Vol. LXIII, 204–211.

[2] H. O. Tejumola (1978) – *Existence of periodic solutions for certain non linear fourth order differential equations*, Paper presented at the Symposium on « Contributions to the development of Mathematics in Nigeria in honour of J. O. C. Ezeilo ».