Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Giuseppe Buttazzo, Gianni Dal Maso

Integral representation on $W^{1,\alpha}(\Omega)$ and BV (Ω) of limits of variational integrals

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **66** (1979), n.5, p. 338–343.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1979_8_66_5_338_0>

Calcolo delle variazioni. — *Integral representation on* $W^{1,\alpha}(\Omega)$ *and* $BV(\Omega)$ *of limits of variational integrals.* Nota di GIUSEPPE BUTTAZZO e GIANNI DAL MASO, presentata (*) dal Corrisp. E. DE GIORGI.

RIASSUNTO. — In questo lavoro vengono studiate le rappresentazioni integrali dei Γ–limiti di successioni di funzionali del calcolo delle variazioni. Notiamo che, se l'interesse principale del lavoro è rivolto ai Γ–limiti di successioni i cui termini non sono tutti eguali, riducendosi a quest'ultimo caso molto particolare si ottengono, come corollario dei risultati di questo lavoro, vari teoremi di semicontinuità di funzionali del tipo $\int_\Omega f(x, u, Du)\,dx$ e di rappresentazione dei prolungamenti di tali funzionali a funzioni appartenenti a $BV(\Omega)$ (funzioni aventi derivate misure).

1. INTRODUCTION

We recall the following characterization of Γ-limits (see [6]): if (X, τ) is a topological space satisfying the first axiom of countability, (F_h) is a sequence of functions from X into $\overline{\mathbf{R}}$ and u is an element of X, then

$$\lambda = \Gamma(\tau^-) \lim_{\substack{h\to\infty \\ v\to u}} F_h(v)$$

if and only if

(*i*) for every sequence (u_h) converging to u in (X, τ)

$$\lambda \leq \liminf_{h\to\infty} F_h(u_h)$$

(*ii*) there exists a sequence (u_h) converging to u in (X, τ) such that

$$\lambda = \lim_{h\to\infty} F_h(u_h)$$

Let Ω be a bounded open subset of $\mathbf{R}^n$, and let (f_h) be a sequence of Borel functions from $\Omega\times\mathbf{R}\times\mathbf{R}^n$ into $\mathbf{R}_+$, with $f_h(x, u, \cdot)$ convex for every $(x, u) \in \Omega\times\mathbf{R}$.

For each open subset A of Ω and $u \in L^1_{loc}(A)$ we set (1)

$$(1.1) \qquad F_h(u, A) = \begin{cases} \int_A f_h(x, u, Du)\,dx & \text{if } u \in C^1(A) \\ +\infty & \text{otherwise.} \end{cases}$$

(*) Nella seduta del 12 maggio 1979.

(1) We obtain the same results if, in (1.1), instead of $C^1(A)$ we take $C^1(A)\cap W^{1,r}(A)$ or $W^{1,r}_{loc}(A)$ (or $C^1(\bar{A})$ if A has a Lipschitz boundary), with $r = \alpha$ in theorem 1 and $r = 1$ in theorems 2 and 3.

The purpose of this paper is to give conditions under which:

(1) there exists $\Gamma\,(L^1_{loc}(A)^-) \lim\limits_{\substack{h\to\infty \\ v\to u}} F_h(v, A) = F(u, A)$;

(2) $F(u, A)$ can be represented as an integral.

This subject has been studied by several authors, e.g. [1], [3], [4], [5], [12], [13], [14], [15], [16], [19], [21].

In theorem 1 we prove that, under suitable growth conditions on $f_h(x, u, p)$ and equi-continuity conditions with respect to the variable u, (1) is verified at least for a subsequence of (F_h), and there exists a function f such that

$$F(u, A) = \int_A f(x, u, Du)\,dx$$

for every $u \in W^{1,\alpha}(A)$ (the space of functions in $L^\alpha(A)$ whose distributional derivatives are in $L^\alpha(A)$).

In theorem 2, by strengthening the continuity of f, and the convergence of (f_h), we obtain an explicit integral representation formula for $F(u, A)$ on $BV(A)$ (the space of functions in $L^1(A)$ whose distributional derivatives are measures with finite variation on A).

In theorem 3 the same formula is obtained without the hypothesis of continuity of $f(x, u, p)$ with respect to the variable u, but assuming that every f_h is positively homogeneous of degree 1 in p.

We note that, in the particular case in which $f_h = f$ for every h, $F(u, A)$ coincides with the greatest lower semicontinuous function $\overline{F}(u, A)$ on $L^1_{loc}(A)$ such that

$$\overline{F}(u, A) \leq \int_A f(x, u, Du)\,dx$$

on $C^1(A)$. Therefore, as a corollary of theorems 1, 2, 3, we get an integral representation formula for $\overline{F}(u, A)$ on $W^{1,\alpha}(A)$ and $BV(A)$. Such a problem has been introduced by J. Serrin in [22], [23], and studied by several authors in [2], [3], [8], [9], [10], [11], [13], [17], [18].

We would like to thank prof. E. De Giorgi for many helpful discussions on this subject.

2. We begin with the following theorem.

THEOREM 1. *Let* $1 \leq \alpha < +\infty$ *and* $1 \leq \beta < +\infty$. *Assume that for all* $h \in \mathbf{N}$, $(x, p) \in \Omega \times \mathbf{R}^n$, $M > 0$

$$(i) \qquad 0 \leq f_h(x, u, p) \leq a_h(x) + c\,|u|^\beta + \Lambda(u)\,|p|^\alpha \qquad \forall u \in \mathbf{R}$$

(*ii*) $|f_h(x, u, p) - f_h(x, v, p)| \leq$
$\leq \rho_{M,h}(x, |u-v|) + \sigma_{M,h}(|u-v|) \cdot f_h(x, u, p) \quad \forall u, v \in [-M, M]$, *where*

$a_h \rightharpoonup a_\infty$ *weakly in* $L^1(\Omega)$,

Λ *is continuous*

$\rho_{M,h}(\cdot, t) \rightharpoonup \rho_{M,\infty}(\cdot, t)$ *weakly in* $L^1(\Omega)$ *for all* t,

$\sigma_{M,h}(t) \to \sigma_{M,\infty}(t)$ *for all* t,

$\rho_{M,h}(x, \cdot)$, $\sigma_{M,h}$ *are nondecreasing functions for a.a.* x

$\lim_{t\to 0} \rho_{M,\infty}(x, t) = \lim_{t \to 0} \sigma_{M,\infty}(t) = 0$ *for a.a.* $x \in \Omega$.

Then there exist an increasing sequence (h_k) *and a function* $f_\infty : \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}_+$, *convex in* p *and satisfying* (*i*) *and* (*ii*) *with* $h = \infty$, *such that for every open subset* A *of* Ω *and every* $u \in W^{1,\alpha}(A) \cap L^\beta(A)$

$$\Gamma(L^1_{loc}(A)^-) \lim_{\substack{k\to\infty \\ v \to u}} F_{h_k}(v, A) = \int_A f_\infty(x, u, Du)\, dx .$$

In the following we denote by $\mathscr{L}^n$ the Lebesgue measure on $\mathbf{R}^n$ and by $\mathscr{H}^m$ the m-dimensional Hausdorff measure (see [7]).

Let A be an open subset of Ω, $u \in BV(A)$ and let μ be the derivative measure of u. If $\mu = \rho + \theta$, with ρ absolutely continuous and θ singular with respect to $\mathscr{L}^n$, then Du denotes the density of ρ with respect to $\mathscr{L}^n$, and M(u) is a Borel subset of A such that $\mathscr{L}^n(M(u)) = |\theta|(A - M(u)) = 0$. Moreover we set

$$u_-(x) = \sup\{t : \{y : u(y) < t\} \text{ has } \mathscr{L}^n\text{-density } 0 \text{ at } x\}$$

$$u_+(x) = \inf\{t : \{y : u(y) > t\} \text{ has } \mathscr{L}^n\text{-density } 0 \text{ at } x\}$$

$$N(u) = \{x \in A : u_-(x) < u_+(x)\}.$$

Let $g : \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}_+$ be a Borel function, with $g(x, u, \cdot)$ convex for each $(x, u) \in \Omega \times \mathbf{R}$. We set, by definition

$$\mathscr{G}(g, u, A) = \int_A g(x, u, Du)\, dx + \int_{M(u)-N(u)} g_0\left(x, u_+, \frac{d\mu}{d|\mu|}\right) d|\mu| +$$

$$+ \int_{N(u)} \left[\frac{1}{u_+(x) - u_-(x)} \int_{u_-(x)}^{u_+(x)} g_0\left(x, s, \frac{d\mu}{d|\mu|}(x)\right) ds\right] d|\mu|(x),$$

where $g_0(x, u, p) = \lim_{t \to 0^+} g(x, u, p/t)\, t$.

LEMMA 1. *Under the previous hypotheses*

$$\mathscr{G}(g, u, A) = \int_{A\times\mathbf{R}} \tilde{g}\left(x, s, \frac{d\alpha}{d|\alpha|}(x, s)\right) d|\alpha|(x, s)$$

where α *is the derivative measure of the characteristic function of the set* $\{(x, s) \in A \times \mathbf{R} : s \leq u(x)\}$ *and* $\tilde{g} : \Omega \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}_- \to \overline{\mathbf{R}}_+$ *is defined by*

$$\tilde{g}(x, u, p, t) = \begin{cases} -g(x, u, -p/t)\, t & \text{if } t < 0 \\ g_0(x, u, p) & \text{if } t = 0. \end{cases}$$

Transforming the functionals F_h, by virtue of the previous lemma, from the Cartesian into the parametric form, and using some results of Yu. G. Reshetnyak [20], we can prove the following theorem.

THEOREM 2. *Assume that* (f_h) *converges pointwise to a function* f *such that*:

(*i*) *for all* $h \in \mathbf{N}, M > 0, (x, u, p) \in \Omega \times [-M, M] \times \mathbf{R}^n$

$$f_h(x, u, p) \geq f(x, u, p) - \varepsilon_{M,h}(x)(1 + |p|)$$

where $\varepsilon_{M,h} \xrightarrow{h} 0$ *in* $L^\infty_{loc}(\Omega)$;

(*ii*) *for* $\mathcal{H}^n$-*a.a.* $(x_0, u_0) \in \Omega \times \mathbf{R}$ *the following continuity condition holds*: $\forall \varepsilon > 0 \quad \exists \delta > 0$ *such that*

$$|f(x, u, p) - f(x_0, u_0, p)| < \varepsilon(1 + |p|)$$

for each (x, u, p) *with* $|x - x_0| < \delta, |u - u_0| < \delta$;

(*iii*) *for all* $h \in \mathbf{N}, (x, u, p) \in \Omega \times \mathbf{R} \times \mathbf{R}^n$

$$\lambda(u)|p| \leq f_h(x, u, p) \leq a(x) + c|u|^\beta + \Lambda(u)|p|$$

with λ, Λ *continuous and strictly positive,*

$$a \in L^\infty_{loc}(\Omega) \cap L^1(\Omega), \qquad 1 \leq \beta < +\infty.$$

Then for every open subset A *of* Ω *and every* $u \in BV(A) \cap L^\beta(A)$

$$\Gamma(L^1_{loc}(A)^-) \lim_{\substack{h \to \infty \\ v \to u}} F_h(v, A) = \mathcal{G}(f, u, A).$$

REMARK 1. Hypothesis (*ii*) in the previous theorem cannot be weakened by the assumption that f is continuous, as the following example shows: $f_h = f$ with

$$f(x, p) = \begin{cases} |p| & \text{if } |p||x|^{1/2} \leq 1 \\ 2|p| - |x|^{-1/2} & \text{if } |p||x|^{1/2} \geq 1 \end{cases}$$

$$u(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} \qquad \Omega =]-1, 1[$$

In this case $\mathcal{G}(f, u, \Omega) = 1$ while $\Gamma(L^1_{loc}(\Omega)^-) \lim_{\substack{h \to \infty \\ v \to u}} F_h(v, \Omega) = 2$.

LEMMA 2. *If* $u \in \mathrm{BV}(\mathrm{A})$, $g : \mathrm{A} \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}_+$ *is a Borel function and* $g(x, u, \cdot)$ *is convex and positively homogeneous of degree* 1 *for each* $(x, u) \in \mathrm{A} \times \mathbf{R}$, *then*

$$\mathcal{G}(g, u, \mathrm{A}) = \int_{\mathbf{R}} \left[\int_{\mathrm{A}} g\left(x, s, \frac{\mathrm{d}\alpha_s}{\mathrm{d}|\alpha_s|}(x)\right) \mathrm{d}|\alpha_s|(x) \right] \mathrm{d}s$$

where α_s *is the derivative measure of the characteristic function of the set* $\{x \in \mathrm{A} : u(x) \geq s\}$.

Using this lemma, we prove the following theorem, in which no assumption is made about the continuity of $u \mapsto f(x, u, p)$.

THEOREM 3. *Assume that for all* $h \in \mathbf{N}$ f_h *is positively homogeneous of degree* 1 *in* p *and that* (f_h) *converges pointwise to a function* f *such that*:

(*i*) *for all* $h \in \mathbf{N}$, $(x, u, p) \in \Omega \times \mathbf{R} \times \mathbf{R}^n$

$$f_h(x, u, p) \geq f(x, u, p) - \varepsilon_h(x, u)\,|p|$$

were $\varepsilon_h(\cdot, u) \to 0$ *in* $\mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$ *for a.a.* $u \in \mathbf{R}$;

(*ii*) *for every* $(u, p) \in \mathbf{R} \times \mathbf{R}^n$ *the function* $x \mapsto f(x, u, p)$ *is continuous* $\mathcal{H}^{n-1}$ — *a.e. in* Ω;

(*iii*) *for each* $(x, u, p) \in \Omega \times \mathbf{R} \times \mathbf{R}^n$

$$\lambda(u)\,|p| \leq f_h(x, u, p) \leq \Lambda(u)\,|p|$$

with Λ *continuous and* $\lambda(u) > 0$ *for a.a.* $u \in \mathbf{R}$.

Then, for each open subset A *of* Ω *and each* $u \in \mathrm{BV}(\mathrm{A})$

$$\Gamma(\mathrm{L}^1_{\mathrm{loc}}(\mathrm{A})^-) \lim_{\substack{h \to \infty \\ v \to u}} \mathrm{F}_h(v, \mathrm{A}) = \mathcal{G}(f, u, \mathrm{A})$$

REMARK 2. If the functions f_h do not depend on x, the hypothesis (*iii*) on the lower bound may be omitted. On the contrary, if every f_h depends on x, this hypothesis cannot be avoided, even if the functions f_h do not depend on u. Indeed, we can give an example in which $f_h = f$ for each h, f is continuous, independent of u, convex and homogeneous of degree 1 in p, $0 \leq f(x, p) \leq |p|$, but the conclusion of the theorem does not hold, even for $u \in \mathrm{C}^1(\mathbf{R}^n)$.

REFERENCES

[1] G. BUTTAZZO and G. DAL MASO (1978) – *Γ-limit of a sequence of non-convex and non-equi-Lipschitz integral functionals.* « Ricerche Mat. », *27*, 235–251.

[2] O. CALIGARIS, F. FERRO and P. OLIVA (1977) – *Sull'esistenza del minimo per problemi di calcolo delle variazioni relativi ad archi di variazione limitata.* « Boll. Un. Mat. Ital. » (5) *14*-B, 340–369.

[3] L. CARBONE and C. SBORDONE – *Some properties of Γ-limits of integral functionals.* « Ann. Mat. Pura Appl. », to appear.

[4] M. CARRIERO and E. PASCALI (1978) – *Γ-convergenza di integrali non negativi maggiorati da funzionali del tipo dell'area.* « Ann. Univ. Ferrara », Sez. VII, *24*, 51–64.

[5] E. DE GIORGI (1975) – *Sulla convergenza di alcune successioni di integrali del tipo dell'area.* « Rend. Mat. », (4) *8*, 277–294.

[6] E. DE GIORGI (1979) – *Convergence problems for functionals and operators*, in Proceedings of the International Meeting on Recent Methods in Non Linear Analysis. Rome, May 8–12, 1978. Bologna.

[7] H. FEDERER (1969) – Geometric Measure Theory, Berlin.

[8] F. FERRO – *Integral characterization of functionals defined on spaces of* BV *functions.* « Rend. Sem. Mat. Univ. Padova », to appear.

[9] N. FUSCO – *Dualità e semicontinuità per integrali del tipo dell'area*, to appear.

[10] M. GIAQUINTA, G. MODICA and J. SOUCEK (1979) – *Functionals with linear growth in the calculus of variations* I. « Comment. Math. Univ. Carolinae », *20*, 143–156.

[11] C. GOFFMAN and J. SERRIN (1964) – *Sublinear functions of measures and variational integrals.* « Duke Math. Journ. », *31*, 159–178.

[12] P. MARCELLINI (1978) – *Periodic solutions and homogenization of non linear variational problems.* « Ann. Mat. Pura Appl. » (4) *117*, 139–152.

[13] P. MARCELLINI (1979) – *Some problems of semicontinuity and of Γ-convergence for integrals of the calculus of variations*, in Proceedings of the International Meeting on Recent Methods in Non Linear Analysis. Rome, May 8–12, 1978. Bologna.

[14] P. MARCELLINI and C. SBORDONE (1977) – *An approach to the asymptotic behaviour of elliptic-parabolic operators.* « J. Math. Pures Appl. » (9) *56*, 157–182.

[15] P. MARCELLINI and C. SBORDONE (1977) – *Dualità e perturbazione di funzionali integrali.* « Ricerche Mat. », *26*, 383–421.

[16] P. MARCELLINI and C. SBORDONE (1978) – *Homogenization of non uniformly elliptic operators.* « Applicable Anal. », *8*, 101–114.

[17] P. MARCELLINI and C. SBORDONE – *Semicontinuity problems in the calculus of variations*, to appear.

[18] C. B. MORREY (1966) – *Multiple Integrals in the Calculus of Variations*, Berlin.

[19] G. MOSCARIELLO (1976) – *Su una convergenza di successioni di integrali del calcolo delle variazioni.* « Atti Accad. Naz. Lincei Rend. Cl. Sci Fis. Mat. Natur. » (8) *61*, 368–375.

[20] YU. G. RESHETNYAK (1968) – *Weak convergence of completely additive vector functions on a set.* « Siberian Math. J. », *9*, 1039–1045.

[21] C. SBORDONE (1975) – *Su alcune applicazioni di un tipo di convergenza variazionale.* « Ann. Scuola Norm. Sup. Pisa Cl. Sci » (4) *2*, 617–638.

[22] J. SERRIN (1959) – *A new definition of the integral for non-parametric problems in the calculus of variations.* « Acta Math. », *102*, 23–32.

[23] J. SERRIN (1961) – *On the definition and properties of certain variational integrals.* « Trans. Amer. Math. Soc. », *101*, 139–167.