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On the argument principle in multidimensional complex manifolds

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 12 maggio 1979

Presiede il Presidente della Classe ANTONIO CARRELLI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *On the argument principle in multidimensional complex manifolds* (*). Nota di GUIDO LUPACCIOLU, presentata (**) dal Socio E. MARTINELLI.

RIASSUNTO. — Si dà un'estensione del classico teorema dell'indicatore logaritmico ("argument principle") al caso in cui l'ambiente sia una varietà complessa multidimensionale e in particolare kähleriana.

§ 1.

In the present work we shall be concerned with holomorphic mappings from a complex manifold X of complex dimension n into a complex space $\mathbf{C}^p$, p being an integer such that $1 \leq p \leq n$.

For each such mapping $\boldsymbol{f}_p = (f^1, \cdots, f^p) : \mathrm{X} \to \mathbf{C}^p$, not identically zero, we denote by $\mathrm{Z}_{\boldsymbol{f}_p}$ the zero set of $f^1, \cdots, f^p$ and by $\omega_{(p,p-1)}(\boldsymbol{f}_p)$ the Martinelli form associated with $\boldsymbol{f}_p$, that is:

$$\omega_{(p,p-1)}(\boldsymbol{f}_p) = \frac{(p-1)!}{(2\pi i)^p} (-1)^{p(p-1)/2} (f^1 \bar{f}^1 + \cdots + f^p \bar{f}^p)^{-p} \, \mathrm{d}f^1 \wedge \cdots \wedge$$

$$\wedge \, \mathrm{d}f^p \wedge \sum_{\alpha=1}^{p} (-1)^{\alpha-1} \mathrm{d}\bar{f}^1 \wedge \cdots \bar{f}^\alpha \, \widehat{\mathrm{d}\bar{f}^\alpha} \cdots \wedge \mathrm{d}\bar{f}^p .$$

(*) Lavoro eseguito dall'Autore come borsista del C.N.R.
(**) Nella seduta del 12 maggio 1979.

This form is smooth and closed outside Z_{f_p}, where it becomes singular.

The present paper deals with an extension to f_p, by means of $\omega_{(p,p-1)}(f_p)$, of the classical argument principle for a function of one complex variable (see [1], p. 151).

The case $p = n$ is merely a generalization of the well known Martinelli integral formula and has already been considered (see [6], chap. II due to E. Martinelli, where one can find also a survey of B. Segre's and Cacciop-poli's contributions about related subjects). The only assumption in this case is that Z_{f_p} must contain at most isolated points. Then the extension of the argument principle to f_n is given by the formula:

$$\int_{\partial D} \varphi\omega_{(n,n-1)}(f_n) = \sum_{z=Z_{f_n}\cap D} \nu(z)\,\varphi(z)\,, \tag{1.1}$$

where D is a relatively compact open domain in X, whose boundary ∂D is almost regular [(1)] and does not intersect Z_{f_n}, φ is any holomorphic function in X and $\nu(z)$ means the multiplicity of z as a common zero of $f^1, \cdots, f^n$ (i.e. as intersection of the analytic varieties $f^1 = 0, \cdots, f^n = 0$ in X). Moreover ∂D is given the orientation induced by the canonical orientation of D (recall that locally, at a point $x \in D$ where $x^h = x^{h'} + ix^{h''}$, $h = 1, \cdots, n$, are complex coordinates, this orientation is given by the differential form $dx^{1'} \wedge dx^{1''} \wedge \cdots \wedge dx^{n'} \wedge dx^{n''}$). We do not dwell here upon the proof of the above formula; for this see [6] [(2)].

Now let us consider the case that $1 \leq p \leq n - 1$. Then the assumption on f_p is that it must be regular at the generic point of Z_{f_p}. More precisely, if C_{f_p} is the critical set of f_p (i.e. the set of the points of X where the differentials $df^1, \cdots, df^p$ are linearly dependent), the following must hold:

(*i*) The analytic set $Z_{f_p} \cap C_{f_p}$ has complex dimension $\leq n - p - 1$ at each point.

This implies that Z_{f_p} is the topological closure of a complex manifold $\hat{Z}_{f_p}$ of complex dimension $n - p$ [(3)]. $\hat{Z}_{f_p}$ is the set of the points of Z_{f_p} where f_p is regular: $\hat{Z}_{f_p} = Z_{f_p} \setminus C_{f_p}$. Therefore if D is any relatively compact open domain in X, the integral of a continuous $2(n-p)$-form in X over $Z_{f_p} \cap D$ may be defined as the integral over $\hat{Z}_{f_p} \cap D$. That this is convergent follows from known results about integration of forms over analytic sets (see [5]). We assume furthermore that the boundary ∂D of D be almost regular and satisfy the following condition:

(*ii*) The set $\hat{Z}_{f_p} \cap \partial D$ has zero measure in $\hat{Z}_{f_p}$.

(1) For a definition see [2], p. 421.

(2) In this book a different agreement is made about orientations so that on the left side of the formula (1.1) there might be a difference about sign.

(3) We are assuming that Z_{f_p} is not empty; otherwise the treatment would be trivial.

Then the extension of f_p to the argument principle is given by the following:

THEOREM. *Under the assumptions (i), (ii) the following formula is valid:*

$$\int_{\partial D} \omega_{(p,p-1)}(f_p) \wedge \varphi_{(n-p,n-p)} = \int_{Z_{f_p} \cap D} \varphi_{(n-p,n-p)}, \tag{1.2}$$

where $\varphi_{(n-p,n-p)}$ *is any bihomogeneous smooth* $\bar\partial$*-closed form in* X *of bidegree* $(n-p, n-p)$.

Note that the $\bar\partial$-closed form $\varphi_{(n-p,n-p)}$ replaces here in a natural way the holomorphic function $\varphi = \varphi_{(0,0)}$ of (1,1), as well as the integral on the right side of (1.2) is a obvious modification of the sum on the right side of (1.1).

Assume, in particular, that X may be given a Kähler metric, with Kähler form Ω. It is known that the exterior power Ω^{n-p} of Ω yields the multiple by $(n-p)!$ of the volume element for the submanifolds of X of complex dimension $n-p$, with respect to the Riemannian metric associated with the given Kähler metric (see [3], p. 143). Therefore, if in (1.2) we choose as $\varphi_{(n-p,n-p)}$ the form $1/(n-p)!\,\Omega^{n-p}$, we get:

COROLLARY I. *If* (X, Ω) *is a Kähler manifold and* f_p, D *satisfy the conditions (i), (ii), the following formula is valid:*

$$\frac{1}{(n-p)!}\int_{\partial D} \omega_{(p,p-1)}(f_p) \wedge \Omega^{n-p} = \mathrm{Vol}\,(Z_{f_p} \cap D), \tag{1.3}$$

where Vol $=$ $(2n-2p)$*-volume.*

Let us consider the case when X is an open subset of $\mathbf{C}^n(x^1, \cdots, x^n)$ and Ω is the standard Kähler form, that is:

$$\Omega = \frac{i}{2}\sum_{h=1}^{n} dx^h \wedge d\bar{x}^h.$$

Then we get:

$$\Omega^{n-p} = (n-p)!\left(\frac{i}{2}\right)^{n-p} \sum_{1 \le h_1 < \cdots < h_{n-p} \le n} dx^{h_1} \wedge d\bar{x}^{h_1} \wedge \cdots \wedge dx^{h_{n-p}} \wedge d\bar{x}^{h_{n-p}},$$

and hence:

COROLLARY II. *If* X *is an open subset of* $\mathbf{C}^n$ *and* f_p, D *satisfy the conditions (i), (ii), the following formula is valid:*

$$\left(\frac{i}{2}\right)^{n-p}\int_{\partial D} \omega_{(p,p-1)}(f_p) \wedge \sum_{1 \le h_1 < \cdots < h_{n-p} \le n} dx^{h_1} \wedge d\bar{x}^{h_1} \wedge \cdots \wedge dx^{h_{n-p}} \wedge d\bar{x}^{h_{n-p}} = \tag{1.4}$$

$$= \mathrm{Vol}\,(Z_{f_p} \cap D).$$

For $p = 1$ this formula has already been considered by W. Wirtinger (see [7]) (4).

§ 2.

In this § we give a proof of the formula (1.2) under a hypothesis on the mapping f_p stronger than (i). We assume that each point of Z_{f_p} be a regular point for f_p, that is:

$$(i') \qquad Z_{f_p} \cap C_{f_p} = \varnothing .$$

In § 3 we shall extend the validity of the formula (1.2) to the case of the weaker assumption (i).

First of all observe that the integral on the left side of (1.2) must be regarded as an improper integral, whose singular set is

$$T = Z_{f_p} \cap \partial D .$$

Therefore it must be evaluated as the limit of the integral

$$(2.1) \qquad \int_{\partial D \setminus T^{[r]}} \omega_{(p,p-1)}(f_p) \wedge \varphi_{(n-p,n-p)}$$

as $r \to \infty$, $\{T^{[r]}\}_{r=0,1,\ldots}$ being a fundamental sequence of open neighbourhoods of T in ∂D. Such a sequence can be easily found by means of the function

$$| f_p | = (f^1 \bar{f}^1 + \cdots + f^p \bar{f}^p)^{1/2} : X \to \mathbf{R}^+ .$$

Since f_p is regular at each point of Z_{f_p} (due to (i')), its image $f_p(X) \subset \mathbf{C}^p$ contains an open neighbourhood of the origin (5). Hence the image $|f_p|(X) \subset \mathbf{R}^+$ of $|f_p|$ contains an open neighbourhood of 0 in $\mathbf{R}^+$. Moreover Sard's theorem (6) yields that "almost all" points of the latter are regular values of $|f_p|$. It follows that we can find a decreasing sequence of regular values of $|f_p|$, say $\{\varepsilon_r\}_{r=0,1,\ldots}$ which converges to 0 and whose first term ε_0 is as small as we please. Then set for each r:

$$X^{[r]} = \{x \in X : |f_p|(x) \le \varepsilon_r\} .$$

(4) In this paper the author assumes the function $f_1 = f$ meromorphic. Then on the right side of the formula there is the difference $\mathrm{Vol}(Z_f \cap D) - \mathrm{Vol}(P_f \cap D)$, where P_f is the polar set of f. However for $p = 1$ the extension from the holomorphic to the meromorphic case can be easily achieved as a consequence of Cousin's theorem (see [6], p. 126).

(5) This is true even if f_p is regular only at some points of Z_{f_p} and hence under the assumption (i).

(6) See [4], p. 10.

It follows from the implicit function theorem that $X^{[r]}$ is a $2n$-dimensional manifold with boundary regularly embedded in X. Let us denote the boundary by $Z^{[r]}$:

$$Z^{[r]} = \{x \in X : |f_p|(x) = \varepsilon_r\}.$$

Now set for each r:

$$T^{[r]} = \mathring{X}^{[r]} \cap \partial D.$$

Next consider for each r the set:

$$D^{[r]} = D \setminus X^{[r]}.$$

If ε_0 is small enough, as we assume, $D^{[r]}$ is a relatively compact open domain in X with almost regular boundary given by:

$$\partial D^{[r]} = (\partial D \setminus T^{[r]}) \cup (-Z^{[r]} \cap D),$$

where the union is disjoint and the negative sign before $Z^{[r]}$ means that the boundary of $X^{[r]}$ must be taken here with the orientation opposite to the canonical one.

Since $\overline{D}^{[r]}$ does not intersect Z_{f_p}, the form $\omega_{(p,p-1)}(f_p) \wedge \varphi_{(n-p,n-p)}$ is not singular there. It is also closed, because $\omega_{(p,p-1)}(f_p)$ is closed and $\varphi_{(n-p,n-p)}$ $\bar{\partial}$-closed, whence:

$$d(\omega_{(p,p-1)}(f_p) \wedge \varphi_{(n-p,n-p)}) = (-1)^{2p-1} \omega_{(p,p-1)}(f_p) \wedge \partial\varphi_{(n-p,n-p)},$$

and the form at the right side has bidegree $(n+1, n-1)$ and therefore is zero. It follows from Stokes' theorem applied to $D^{[r]}$ that:

$$(2.2) \qquad \int_{\partial D \setminus T^{[r]}} \omega_{(p,p-1)}(f_p) \wedge \varphi_{(n-p,n-p)} = \int_{Z^{[r]} \cap D} \omega_{(p,p-1)}(f_p) \wedge \varphi_{(n-p,n-p)}.$$

Now recall the assumption (i'). Since $\overline{D}$ is compact, if ε_0 is small enough, as we assume, we can find relatively compact open sets $U_1, \cdots, U_N$ in X such that:

(a) $$Z^{[0]} \cap \overline{D} \subset U = \bigcup_{k=1}^{N} U_k,$$

(b) For each $k = 1, \cdots, N$ there exist $n-p$ holomorphic functions in $U_k, g_k^1, \cdots, g_k^{n-p}$, such that the mapping

$$c_k = (f^1, \cdots, f^p, g_k^1, \cdots, g_k^{n-p}) : U_k \to c_k(U_k) \subset \mathbf{C}^n$$

is a local coordinate system on X.

Therefore, if we consider a smooth partition of unity on U, $\sum_{k=1}^{N} \gamma_k = 1$, such that supp $\gamma_k \subset U_k$, we may replace the integral on the right side of (2.2) by the sum $\sum_{k=1}^{N} \mathcal{T}_k^{[r]}$, where

$$\mathcal{T}_k^{[r]} = \int_{c_k(U_k \cap Z^{[r]} \cap D)} c_k^{-1*} \left(\omega_{(p,p-1)}(f_p) \wedge \gamma_k \varphi_{(n-p,n-p)}\right). \tag{2.3}$$

Hence, to conclude the proof, we have to show that $\mathcal{T}_k^{[r]}$ tends to the limit

$$\mathcal{T}_k = \int_{c_k(U_k \cap Z_{f_p} \cap D)} c_k^{-1*} \left(\gamma_k \varphi_{(n-p,n-p)}\right) \tag{2.4}$$

as $r \to \infty$.

First note that:

$$c_k^{-1*} \omega_{(p,p-1)}(f_p) = \omega_{(p,p-1)}(x^1, \cdots, x^p).$$

Moreover, if E denotes the linear subspace of $\mathbf{C}^n$ represented by the equations $x^1 = \cdots = x^p = 0$ and $E^{[r]}$ the hypersurface represented by the equation $x^1 \bar{x}^1 + \cdots + x^p \bar{x}^p = \varepsilon_r^2$, then:

$$c_k(U_k \cap Z_{f_p} \cap D) = c_k(U_k \cap D) \cap E, \quad c_k(U_k \cap Z^{[r]} \cap D) = c_k(U_k \cap D) \cap E^{[r]}.$$

Therefore, setting

$$\psi_{k(n-p,n-p)} \begin{cases} = c_k^{-1*}(\gamma_k \varphi_{(n-p,n-p)}) & \text{in} \quad c_k(U_k \cap D), \\ = 0 & \text{in} \quad \mathbf{C}^n \setminus c_k(U_k \cap D), \end{cases}$$

we get:

$$\mathcal{T}_k^{[r]} = \int_{E^{[r]}} \omega_{(p,p-1)}(x^1, \cdots, x^p) \wedge \psi_{k(n-p,n-p)}, \quad \mathcal{T}_k = \int_E \psi_{k(n-p,n-p)}. \tag{2.5}$$

Now observe that $E^{[r]}$ may be regarded as the product $S_{\varepsilon_r} \times E$, where S_{ε_r} is the $(2p-1)$-sphere with center at the origin and radius ε_r in the linear subspace of $\mathbf{C}^n$ represented by the equations $x^{p+1} = \cdots = x^p = 0$. Therefore, if S denotes the unit sphere in this subspace, we may perform the change of variables

$$\sigma_r : (x^1, \cdots, x^n) \mapsto (\varepsilon_r x^1, \cdots, \varepsilon_r x^p, x^{p+1}, \cdots, x^n)$$

and replace $E^{[r]}$ by $S \times E$ in $\mathcal{T}_k^{[r]}$. Since the form $\omega_{(p,p-1)}(x^1, \cdots, x^p)$ is invariant under this change of variables, we get:

$$\mathcal{T}_k^{[r]} = \int_{S \times E} \omega_{(p,p-1)}(x^1, \cdots, x^p) \wedge \sigma_r^* \psi_{k(n-p,n-p)}. \tag{2.6}$$

Now let us show that:

$$\lim_{r\to\infty} \sigma_r^* \psi_{k(n-p,n-p)} = \psi_{k(n-p,n-p)} \mid E \ ^{(7)}$$

almost everywhere on $S \times E$, uniformly on the compact subsets where the limit exists. As a consequence we shall be allowed to evaluate the limit of $\mathcal{T}_k^{[r]}$ when $r \to \infty$ under the sign of integration, getting:

$$\lim_{r\to\infty} \mathcal{T}_k^{[r]} = \int_{S\times E} \omega_{(p,p-1)}(x^1, \cdots, x^p) \wedge (\psi_{k(n-p,n-p)} \mid E) . \tag{2.7}$$

Consider the set $E_0 = E \cap c_k(\operatorname{supp} \gamma_k \cap \partial D)$. It follows from the assumption *(ii)* that E_0 has zero measure in E. Hence $S \times E_0$ has zero measure in $S \times E$. For each $\boldsymbol{x} = (x^1, \cdots, x^n)$ in $S \times E \setminus S \times E_0$ consider the projection $\boldsymbol{x}' = (0, \cdots, 0, x^{p+1}, \cdots, x^n)$ of $\boldsymbol{x}$ onto E. Let $\delta(\boldsymbol{x})$ be the distance of $\boldsymbol{x}'$ from $c_k(\operatorname{supp} \gamma_k \cap \partial D)$. This is positive because $\boldsymbol{x}' \notin E_0$ and $c_k(\operatorname{supp} \gamma_k \cap \partial D)$ is compact. Therefore there exists an integer r_0 such that $\varepsilon_r < \delta(\boldsymbol{x})$ for $r \geq r_0$ and consequently $\sigma_r(\boldsymbol{x}) \notin c_k(\operatorname{supp} \gamma_k \cap \partial D)$. Since the form $\psi_{k(n-p,n-p)}$ is continuous outside $c_k(\operatorname{supp} \gamma_k \cap \partial D)$, it follows that $\sigma_r^* \psi_{k(n-p,n-p)}$ is continuous at the point $\boldsymbol{x}$ for $r \geq r_0$. Therefore the limit $\lim_{r\to\infty} \sigma_r^* \psi_{k(n-p,n-p)}$ exists at each point $\boldsymbol{x} \in S \times E \setminus S \times E_0$ and may be obtained by replacing ε_r by 0 in $\sigma_r^* \psi_{k(n-p,n-p)}$, which obviously yields the form $\psi_{k(n-p,n-p)} \mid E$.

Moreover that this limit is uniform on compact subsets of $S \times E \setminus S \times E_0$ follows easily from the fact that the function δ is continuous and positive.

Finally consider the integral on the right side of (2.7). It may be computed as the product of the two integrals

$$\int_S \omega_{(p,p-1)}(x^1, \cdots, x^p) \ , \ \mathcal{T}_k .$$

Since the first is 1 [(8)], the proof is completed.

§ 3.

Now assume that $\boldsymbol{f}_p$ satisfy the condition *(i)*, but not the stronger condition *(i')*.

As we have already noticed (see § 2, footnote 5), the image $\boldsymbol{f}_p(X) \subset \mathbf{C}^p$ of $\boldsymbol{f}_p$ contains an open neighbourhood of the origin, say I. Then let I' denote

(7) Obviously the restriction of $\psi_{k(n-p,n-p)}$ to E is obtained by putting 0 instead of $x^h, \bar{x}^h, dx^h, d\bar{x}^h$ for $h = 1, \cdots, p$, and may be regarded as a form on the whole $\mathbf{C}^n$.

(8) To see this write the formula (1.1) for $n = p$, $X = \mathbf{C}^p$, $\boldsymbol{f}_p = (x^1, \cdots, x^p)$, $\varphi \equiv 1$, $\partial D = S$.

the set of critical values of f_p in I and I″ the set of those regular values $\lambda = (\lambda', \cdots, \lambda^p)$ in I for which $Z_{f_p - \lambda} \cap \partial D$ has not zero measure in $Z_{f_p-\lambda}$. It follows from Sard's theorem that I′ has zero measure, and it can be shown that the same is true for I″, although we do not linger over this point. Hence the set $J = I \setminus (I' \cup I'')$ is everywhere dense in I, so that we can find a sequence $\{\lambda^{[s]}\}_{s=1,2\cdots}$ of points of J converging to the origin as $s \to \infty$. Then set for each s:

$$f_p^{[s]} = f_p - \lambda^{[s]} .$$

Since $f_p^{[s]}$ satisfies the conditions (i'), (ii), the formula (1.2) is valid for $f_p^{[s]}$, as we have proved in § 2. Hence, taking the limit as $s \to \infty$, it follows that this is valid also for f_p.

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