Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

# Rendiconti

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# Comparison theorems for a coupled system of singular hyperbolic differential inequalities. I. Time-independent uncoupling coefficients

*Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8*, Vol. **66** (1979), n.4, p. 250–254.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1979_8_66_4_250_0>

**Equazioni a derivate parziali.** — *Comparison theorems for a coupled system of singular hyperbolic differential inequalities*. I. *Time-independent uncoupling coefficients*. Nota di C. Y. CHAN e EUTIQUIO C. YOUNG, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — L'articolo contiene teoremi di confronto per un sistema accoppiato di disequazioni differenziali iperboliche singolari nel caso in cui i coefficienti di accoppiamento non dipendono dal tempo.

## 1. INTRODUCTION

Comparison theorems of Sturm's type for partial differential inequalities of the elliptic type have been studied extensively. See, for example, Chan [1], Chan and Young [2], Diaz and McLaughlin [3], Dunninger and Weinacht [4], Kuks [5], Noussair and Swanson [6], Swanson [7], and Yoshida [8]. For parabolic type, we refer to the work of Chan and Young [9–10], Dunninger [11], and Noussair and Swanson [6]. Comparison theorems for hyperbolic equations were established by Kreith [12–13], Travis [14], and Young [15]. Recently, Travis and Young [16] obtained comparison theorems for ultra-hyperbolic equations.

The purpose here is to establish some comparison results for a coupled system of hyperbolic differential inequalities subject to first boundary conditions. We should point out that although the paper recovers some of the results of Young [15] in the case of an uncoupled system of singular hyperbolic equations, the method of proofs employed here deviates from that used in that paper, especially for $k < 0$. In particular, no reference here is made to Bessel equation and the properties of Bessel function. This is due to the fact that our present system involves inequalities. Coupled systems of hyperbolic differential equations occur, for example, in describing the potentials in the propagation of electromagnetic waves along two transmission lines (cf. Kuznetsov and Stratonovich [17, p. 92]).

## 2. COMPARISON THEOREMS

Let D be a bounded domain in the real n-dimensional Euclidean space with sufficiently smooth boundary $\partial D$, $R = D \times (0, T)$ with $T > \infty$, $R^-$ be the closure of R, $x = (x_1, x_2, x_3, \cdots, x_n)$ in D, and $S = \partial D \times (0, T)$. Let us

(*) Nella seduta del 21 aprile 1979.

consider the coupled system of hyperbolic differential inequalities

$$u_{tt} + \frac{k}{t} u_t - [a_{ij}(x) u_{x_i}]_{x_j} + b(x) u - c(x, t) v \geq 0 \quad \text{in } R, \tag{2.1}$$

$$v_{tt} + \frac{k}{t} v_t - [A_{ij}(x) v_{x_i}]_{x_j} + B(x) v + C(x, t) u \leq 0 \quad \text{in } R, \tag{2.2}$$

subject to the lateral boundary conditions

$$u = 0 = v \quad \text{on } S, \tag{2.3}$$

where $k$ is a real parameter such that $-\infty < k < \infty$, and the repeated indices are to be summed from one to $n$. It will be seen that the parameter $k$ plays a significant role in our results.

We assume that the coefficient matrices $(a_{ij})$ and $(A_{ij})$ are symmetric, positive definite and in class $C^1(R^-)$, and the functions $b$, $c$, B and C belong to class $C(R^-)$. A solution $(u, v)$ of the coupled system (2.1) and (2.2) belongs to $C^2(R) \cap C^1(R^-)$.

The following lemma, which is needed later on, may be established by using the method of Fox [18] for our domain R.

LEMMA 1. *If* $(u, v)$ *is a solution of* (2.1) *and* (2.2), *then for any* $k \neq 0$, $u_t(x, 0) = 0 = v_t(x, 0)$.

In this paper, we assume that the functions $b$, $c$ and C are nonnegative,

$$a_{ij} \leq A_{ij} \quad \text{for } i, j = 1, 2, 3, \cdots, n,$$
$$b \leq B,$$

where at least one strict inequality holds somewhere in D, and the first derivatives of $a_{ij}$ and $A_{ij}$, $b$ and B are Hölder continuous on $D^-$. We shall also need the following condition:

(I) $u > 0$ in R; for $x \in D$, $u(x, T) = 0$, and if $k \leq 0$, then in addition $u(x, 0) = 0$; $v(x_0, t_0) > 0$ for some point $(x_0, t_0)$ in R.

THEOREM 1. *For* $k \geq 0$, *if* $(u, v)$ *is a solution of* (2.1), (2.2) *and* (2.3) *such that condition* (I) *holds, then* $v$ *must vanish somewhere in* R.

*Proof.* Let $\phi$ and $\psi$ be the normalized eigenfunctions corresponding to the first eigenvalues $\lambda_0$ and $\mu_0$ of the following problems

$$-(a_{ij}\phi_{x_i})_{x_j} + b\phi = \lambda\phi \quad \text{in } D, \quad \phi = 0 \quad \text{on } \partial D, \tag{2.4}$$

$$-(A_{ij}\psi_{x_i})_{x_j} + B\psi = \mu\psi \quad \text{in } D, \quad \psi = 0 \quad \text{on } \partial D, \tag{2.5}$$

respectively. Under the assumptions made on the coefficients of (2.4) and (2.5), it follows from Krasnosel'skii [19, p. 259] that $\phi$ and $\psi$ are nonnegative, and $\lambda_0$ and $\mu_0$ are simple and positive. By using the variational principle, $\lambda_0 < \mu_0$.

Suppose that $v > 0$ in R. Let us define

$$\mathrm{U}(t) = \int_{\mathrm{D}} u(x, t)\, \phi(x)\, \mathrm{d}x ,$$

$$\mathrm{V}(t) = \int_{\mathrm{D}} v(x, t)\, \psi(x)\, \mathrm{d}x .$$

It follows from our assumptions that for $0 < t < \mathrm{T}$, $\mathrm{U}(t)$ and $\mathrm{V}(t)$ are positive, $\mathrm{U}(\mathrm{T}) = 0$, and for $k = 0$, we also have $\mathrm{U}(0) = 0$. Let

$$f(t) = \int_{\mathrm{D}} c(x, t)\, v(x, t)\, \phi(x)\, \mathrm{d}x ,$$

$$g(t) = \int_{\mathrm{D}} \mathrm{C}(x, t)\, u(x, t)\, \psi(x)\, \mathrm{d}x .$$

Using (2.1), (2.2), (2.3), (2.4), (2.5) and the divergence theorem, we have

$$\mathrm{U}''(t) + \frac{k}{t}\mathrm{U}'(t) + \lambda_0 \mathrm{U}(t) \geq f(t) \geq 0 ,$$

$$\mathrm{V}''(t) + \frac{k}{t}\mathrm{V}'(t) + \mu_0 \mathrm{V}(t) \leq -g(t) \leq 0 .$$

Let us first consider the case $k > 0$. Integrating the inequality

$$[t^k (\mathrm{U}'\mathrm{V} - \mathrm{U}\mathrm{V}')]' + (\lambda_0 - \mu_0)\, t^k \mathrm{U}\mathrm{V} \geq t^k (\mathrm{V}f + \mathrm{U}g) \tag{2.6}$$

over $(0, \mathrm{T})$, and noting that $u(\mathrm{T}) = 0$, we obtain

$$\mathrm{T}^k \mathrm{U}'(\mathrm{T})\, \mathrm{V}(\mathrm{T}) + (\lambda_0 - \mu_0) \int_0^{\mathrm{T}} t^k\, \mathrm{U}(t)\, \mathrm{V}(t)\, \mathrm{d}t \geq \int_0^{\mathrm{T}} t^k [\mathrm{V}(t) f(t) + \mathrm{U}(t) g(t)]\, \mathrm{d}t . \tag{2.7}$$

Since $\mathrm{U} > 0$ for $0 < t < \mathrm{T}$, and $\mathrm{U}(\mathrm{T}) = 0$, it follows that $\mathrm{U}'(\mathrm{T}) \leq 0$. Because $\lambda_0 < \mu_0$, the left-hand side of (2.7) is negative, but the right-hand side is nonnegative. Thus we have a contradiction, and so $v$ must vanish in R.

When $k = 0$, the integration of (2.6) adds the extra term $-\mathrm{U}'(0)\, \mathrm{V}(0)$ to the left-hand side of (2.7). The additional condition that $u(x, 0) = 0$ for any $x$ in D implies that $\mathrm{U}'(0) \geq 0$ since $\mathrm{U}(t) > 0$ for $t > 0$. Hence the left-hand side of (2.7) is still negative and we again have a contradiction. Thus the theorem is proved.

When $k < 0$, we have a weaker result stated below. Let $\mathrm{R}^* = \mathrm{D} \times [0, \mathrm{T})$.

THEOREM 2. *For* $k < 0$, *if* $(u, v)$ *is a solution of* (2.1), (2.2) *and* (2.3) *such that condition* (I) *holds, then* $v$ *must vanish somewhere in* $R^*$.

*Proof.* Let us assume that $v > 0$ in $R^*$. Since $k < 0$, the integrals of the inequality (2.6) over $(0, T)$ may no longer exist. Let us rewrite (2.6) as

$$\text{(2.8)} \qquad U''V - UV'' + \frac{k}{t}(U'V - UV') + (\lambda_0 - \mu_0)UV \geq Vf + Ug.$$

Let $W = U'V - UV'$, and $G = (\mu_0 - \lambda_0)UV + Vf + Ug$. Then (2.8) becomes

$$W' + \frac{k}{t}W \geq G.$$

By Lemma 1, $U'(0)$ is zero while it follows from condition (I) that $U(0)$ and $U(T)$ are zero. Hence $U'(T) \leq 0$, $W(0) = 0$, and $W(T) \leq 0$. Let

$$W' + \frac{k}{t}W = G^*$$

Then,

$$W(t) = t^{-k}\left[-\int_t^T \tau^k G^*(\tau)\, d\tau + W(T)T^k\right]$$

$$\leq t^{-k}\left[-\int_t^T \tau^k G(\tau)\, d\tau + W(T)T^k\right].$$

Since $W(T) \leq 0$ and $G(t) > 0$, we have $W(t) < 0$ for $0 < t < T$. But $W = V^2(U/V)'$, it follows that $U/V$ is a decreasing function of $t$. Now $u(x, 0) = 0$ gives $U(0) = 0$, which implies that $U/V$ is zero for $t = 0$. This in turn gives $U/V < 0$ for $t > 0$, and we have a contradiction since both $U$ and $V$ are positive for $0 < t < T$. Thus $v$ must vanish somewhere in $R^*$.

By using a proof similar to that of Theorem 2 of Young [15], the following stronger result can also be established for $k < 0$ if we consider the corresponding problem involving (2.2) and the adjoint operator of (2.1):

$$\text{(2.9)} \qquad u_{tt} - \left(\frac{k}{t}u\right)_t - [a_{ij}(x)\, u_{x_i}]_{x_j} + b(x)\, u - c(x, t)\, v \geq 0 \qquad \text{in } R,$$

subject to the lateral boundary conditions (2.3).

THEOREM 3. *For* $k < 0$, *if* $(u, v)$ *is a solution of* (2.9), (2.2) *and* (2.3) *such that condition* (I) *holds, then* $v$ *must vanish somewhere in* $R$.

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