Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

H.O. Tejumola

Periodic Solutions of certain third order differential equations

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **66** (1979), n.4, p. 243–249.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1979_8_66_4_243_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1979.

Equazioni differenziali ordinarie. — *Periodic Solutions of certain third order differential equations.* Nota di H. O. TEJUMOLA, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si dimostrano due teoremi di esistenza di soluzioni periodiche per una classe di equazioni $\dddot{x} + f(\dot{x})\ddot{x} + g(x)\dot{x} + h(x) = p(t, x, \dot{x}, \ddot{x})$.

1. We shall be concerned here with equazions of the form

$$\dddot{x} + f(\dot{x})\ddot{x} + g(x)\dot{x} + h(x) = p(t, x, \dot{x}, \ddot{x}), \tag{1.1}$$

where the functions f, g, h and p are continuous functions of the arguments shown, and p is ω-periodic in t ($\omega > 0$). In the special case

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx = p(t), \tag{1.2}$$

with a, b, c all constants and p ω-periodic in t, it is well known that (1.2) admits of a unique ω-periodic solution provided one of the following conditions holds:

$$ac < 0, \qquad b \text{ arbitrary}, \tag{1.3}$$

$$ac > 0 \quad \text{and} \quad a^{-1}c \neq 4\pi^2\omega^{-2}, \qquad b \text{ arbitrary}, \tag{1.4}$$

$$b \neq 4\pi^2\omega^{-2}, \qquad a, c \text{ arbitrary}. \tag{1.5}$$

Recently, Ezeilo [1, Theorems 1, 2, 3] showed that the conditions (1.3) and (1.4) for existence of ω-periodic solutions can be extended to the more general equation (1.1). Our interest here centres round the alternative condition (1.5), and we shall show that suitable extensions of this are also available for the equation (1.1).

THEOREM 1. *Suppose there exist constants* $A > 0, b$ *with*

$$b < 4\pi^2\omega^{-2} \tag{1.6}$$

such that g, h and p respectively satisfy

$$g(x) \leq b \quad \textit{for all } x, \tag{1.7}$$

$$h(x)\operatorname{sgn} x \geq A \quad (|x| \geq 1), \tag{1.8}$$

$$|p(t, x, y, z)| \leq A \quad \textit{for all } t, x, y, z. \tag{1.9}$$

Then the equation (1.1) *has at least one ω-periodic solution for all arbitrary continuous function f.*

(*) Nella seduta del 21 aprile 1979.

The condition (1.8) corresponds to the situation $c > 0$ in (1.5). We also have, in the other direction; that is, for $c < 0$ (1.5), that

THEOREM 2. *Suppose that there exists a positive constant* c *such that*

$$x^{-1} h(x) \leq -c \qquad (|x| \geq 1) \tag{1.10}$$

and that g *and* p *satisfy the conditions* (1.7) *and* (1.9) *in Theorem* 1, *with* b *subject to* (1.6). *Then* (1.1) *has at least one* ω*-periodic solution for all arbitrary continuous functions* f.

It is of interest to compare the above results with Ezeilo's [1, Theorem 4] for the special case

$$\dddot{x} + a\ddot{x} + g(x)\dot{x} + h(x) = p(t, x, \dot{x}, \ddot{x})$$

with $a > 0$ a constant, g and h satisfying

$$g(x) \leq b, \qquad xh(x) \geq cx^2 - d \qquad \text{for all } x, \tag{1.11}$$

where $b > 0$, $c > 0$, $d \geq 0$ are constants and

$$ab < c. \tag{1.12}$$

The condition (1.12), with a, b, c positive and $a^{-1} c = 4\pi^2 \omega^{-2}$ implies (1.7). What is more, in Theorems 1 and 2 above, the function f is arbitrary, the constant b satisfying (1.6) need not be positive and in contrast with (1.11), $h(x)$ need not take the sign of x. It must however be conceded that the condition on p in [1, Theorem 4] is weaker than the corresponding one, (1.9) here.

Our generalization here of (1.5) may seem a partial one, since it concerns only the case $b < 4\pi^2 \omega^{-2}$ and not

$$b > 4\pi^2 \omega^{-2}. \tag{1.13}$$

It turns out however that the situation considered here is the only outstanding interesting case. For, if (1.13) holds, then the resulting equation is either of the Routh–Hurwitz type (cf [2]) or is covered by the results in [1] and [2].

2. *Notations*. In what follows the letters D, D_1, D_2, with or without suffixes denote finite positive constants whose magnitudes depend only on the constants b, c, ω and A as well as on the functions f, g and h. The Ds with suffixes, $D_1, D_2, \cdots$, retain a fixed identity throughout, while those without suffixes are not necessarily the same each time they occur.

3. *Proof of Theorem 1*. The procedure here is by the Leray-Sehauder fixed point technique. Consider, instead of (1.1), the parameter λ-dependent equation

$$\dddot{x} + \lambda f(\dot{x})\ddot{x} + (1-\lambda) b\dot{x} + \lambda g(x)\dot{x} + (1-\lambda) cx + \lambda h(x) = \\ = \lambda p(t, x, \dot{x}, \ddot{x}), \tag{3.1}$$

where c is an arbitrarily chosen, but fixed, positive constant and $0 \leq \lambda \leq 1$. Observe that (3.1) reduces to the original equation (1.1) when $\lambda = 1$ and to the constant—co-efficient equation

$$\dddot{x} + b\dot{x} + cx = 0 \tag{3.2}$$

when $\lambda = 0$. With b subject to the condition (1.6), the equation (3.2) does not admit of non-trivial ω-periodic solutions. Therefore, following the arguments in [1; § 4], it will suffice here to prove the existence of an a-priori bound:

$$\max_{0 \leq t \leq \omega} (| x(t) | + | \dot{x}(t) | + | \ddot{x}(t) |) \leq D \tag{3.3}$$

for every ω-periodic solution of (3.1), with $\lambda \in (0, 1)$.

To establish (3.3), let $x = x(t)$ be an arbitrary ω-periodic solution of (3.1), with $0 < \lambda < 1$. Multiply (3.1) all thorugh by $\dot{x}$ and integrate between τ and $\tau + \omega$. We have, since x is ω-periodic and g and h satisfy (1.7) and (1.8) respectively, that

$$\int_\tau^{\tau+\omega} \ddot{x}^2 \, dt - b \int_\tau^{\tau+\omega} \dot{x}^2 \, dt \leq A \int_\tau^{\tau+\omega} | \dot{x} | \, dt .$$

But the ω-periodicity of x also implies that (cf [1; § 5])

$$4 \pi^2 \omega^{-2} \int_\tau^{\tau+\omega} \dot{x}^2 \, dt \leq \int_\tau^{\tau+\omega} \ddot{x}^2 \, dt . \tag{3.4}$$

Therefore,

$$\left(1 - \frac{b}{4} \pi^{-2} \omega^2\right) \int_\tau^{\tau+\omega} \ddot{x}^2 \, dt \leq A \int_\tau^{\tau+\omega} | \dot{x} | \, dt$$

$$\leq A \left(\int_\tau^{\tau+\omega} \dot{x}^2 \, dt \right)^{1/2}$$

$$\leq \tfrac{1}{2} A \pi^{-1} \omega^{3/2} \left(\int_\tau^{\tau+\omega} \ddot{x}^2 \, dt \right)^{1/2}$$

by Schwarz's inequality and (3.4). Since $b < 4 \pi^2 \omega^{-2}$, the last inequality shows that

$$\int_\tau^{\tau+\omega} \ddot{x}^2 \, dt \leq D_1 \tag{3.5}$$

and in view of (3.4), we also have that

$$\int_{\tau}^{\tau+\omega} \ddot{x}^2 \, dt \leq D_2 \tag{3.6}$$

for some constants D_1, D_2. Now since $x(0) = x(\omega)$, there exists $\tau_0 \in [0, \omega]$ such that $\dot{x}(\tau_0) = 0$. Thus

$$\dot{x}(t) = \int_{\tau_0}^{\tau} \ddot{x}(s) \, ds$$

and so,

$$\max_{0 \leq t \leq \omega} |\dot{x}(t)| \leq \int_{\tau}^{\tau_0+\omega} |\ddot{x}(s)| \, ds$$

$$\leq \omega^{1/2} \left(\int_{\tau}^{\tau+\omega} \ddot{x}^2(s) \, ds \right)^{1/2}.$$

By (3.5), this implies that

$$\max_{0 \leq t \leq \omega} |\dot{x}(t)| \leq D . \tag{3.7}$$

To obtain a bound for $|x(t)|$, integrate (3.1) with respect to t to yield

$$\int_{\tau}^{\tau+\omega} \{(1-\lambda) cx + \lambda [h(x) - p(t, x, \dot{x}, \ddot{x})]\} \, dt = 0 . \tag{3.8}$$

In view of (1.8) and (1.9) it follows that $|x(\tau_1)| \leq 1$ for some τ_1. For otherwise, $x(t) \geq 1$ for all t or $x(t) \leq -1$, so that the left hand side of (3.8) is either strictly positive or negative. From the identity

$$x(t) = x(\tau_1) + \int_{\tau_1}^{t} \dot{x}(s) \, ds ,$$

we have that

$$\max_{0 \leq t \leq \omega} |x(t)| \leq 1 + \int_{\tau_1}^{\tau_1+\omega} |\dot{x}(s)| \, ds$$

$$\leq 1 + \omega^{1/2} \left(\int_{\tau_1}^{\tau_1+\omega} \dot{x}^2(s) \, ds \right)^{1/2},$$

and hence,

$$\max_{0 \leq t \leq \omega} |x(t)| \leq D , \tag{3.8}$$

by (3.6).

We now turn to the bound for $|\ddot{x}(t)|$. Multiplying (3.1) by $\dddot{x}$ and integrating we obtain

$$\int_{\tau}^{\tau+\omega} \dddot{x}^2 \, dt = -\lambda \int_{\tau}^{\tau+\omega} f(\dot{x}) \, \ddot{x} \, \dddot{x} \, dt + \int_{\tau}^{\tau+\omega} \dddot{x} \, Q \, dt , \tag{3.9}$$

where

$$Q = \lambda p (t , x , \dot{x} , \ddot{x}) - (1 - \lambda) \, b\dot{x} - \lambda g (x) \, \dot{x} - (1 - \lambda) \, cx - \lambda h (x)$$

satisfies

$$|Q| \leq D_3 , \tag{3.10}$$

by (1.9), (3.7) and (3.8). But by (3.7), $|f(\dot{x})| \leq D_4$, so that

$$\left| -\lambda \int_{\tau}^{\tau+\omega} f(\dot{x}) \, \ddot{x} \, \dddot{x} \, dt \right| \leq D_5 \int_{\tau}^{\tau+\omega} |\ddot{x}| \, |\dddot{x}| \, dt$$

$$\leq D_6 \left(\int_{\tau}^{\tau+\omega} \ddot{x}^2 \, dt \right)^{1/2} \left(\int_{\tau}^{\tau+\omega} \dddot{x}^2 \, dt \right)^{1/2}$$

$$\leq D_6 \left(\int_{\tau}^{\tau+\omega} \dddot{x}^2 \, dt \right)^{1/2} ,$$

by (3.5). Thus from (3.9) and (3.10), we have that

$$\int_{\tau}^{\tau+\omega} \dddot{x}^2 \, dt \leq D_7 \left(\int_{\tau}^{\tau+\omega} \dddot{x}^2 \, dt \right)^{1/2} ,$$

and hence that

$$\int_{\tau+\omega}^{\tau} \dddot{x}^2 \, dt \leq D_8$$

for some D_8. The result

$$\max_{0 \leq t \leq \omega} |\ddot{x}(t)| \leq D$$

now follows readily. This completes the verification of (3.3), and also the proof of Theorem 1.

4. *Proof of Theorem 2.* The procedure is exactly the same as in § 3 except for the modications which we now point out. In order to utilize hypothesis (1.10), consider, instead of (3.1), the λ-dependent equation

$$\dddot{x} + \lambda f (\dot{x}) \, \ddot{x} + (1 - \lambda) \, b\dot{x} + \lambda g (x) \, \dot{x} - (1 - \lambda) \, cx + \lambda h (x) = \tag{4.1}$$

$$= \lambda p (t , x , \dot{x} , \ddot{x}) ,$$

where $c > 0$ is an arbitrarily chosen but fixed constant and, as before, $0 \leq \lambda \leq 1$. Note that (4.1) reduces to the original equation (1.1) when $\lambda = 1$, and for $\lambda = 0$, it reduces to the linear equation

$$\ddot{x} + b\dot{x} - cx = 0 \tag{4.2}$$

Since b is subject to the condition (1.6), it is easy to see that the equation (4.2) does not admit of a non-trivial ω-periodic solution.

The estimates (3.5), (3.6) and (3.7) can now be obtained as in § 3 for any ω-periodic solution $x(t)$ of (4.1). In fact, the only other modification is in the verification of $\max_{0 \leq t \leq \omega} |x(t)| \leq D$. Integrating (4.1) with respect to t, and using the ω-periodicity of $x = x(t)$, we have analogous to (3.8), that

$$\int_{\tau}^{\tau+\omega} \{h_\lambda(x) - \lambda p(t, x, \dot{x}, \ddot{x})\} \, dt = 0, \tag{4.3}$$

where

$$h_\lambda(x) = -(1 - \lambda) cx + \lambda h(x), \qquad 0 < \lambda < 1.$$

Note from (1.10) that

$$x^{-1} h_\lambda(x) \leq -c \,(|x| \geq 1), \qquad 0 < \lambda < 1,$$

so that

$$h_\lambda(x) \leq -cx \qquad \text{if} \quad x \geq 1, \tag{4.4}$$

and

$$h_\lambda(x) \geq -cx \qquad \text{if} \quad x \leq -1. \tag{4.5}$$

Thus, so long as $0 < \lambda < 1$ (4.3) and (1.9) imply that there is a $D_9 > 1$ such that

$$|x(T)| < D_9 \qquad \text{for some} \quad T. \tag{4.6}$$

For otherwise, we have that either $x(t) \geq D_9 > 1$ for all t or $x(t) \leq -D_9 < -1$ for all t. In the former case, the left hand side of (4.3) will, in view of (4.4), be strictly negative if $x(t) > 1$ is large enough and, in the latter, strictly positive by (4.5), if $x(t) < -1$ is large enough. Thus (4.6) holds. The estimate

$$\max_{0 \leq t \leq \omega} |x(t)| \leq D$$

will now follow in the usual way from the identity:

$$x(t) = x(T) + \int_{T}^{t} \dot{x}(s) \, ds$$

and the estimates (4.6) and (3.6). The rest of the proof of the boundedness of $|\ddot{x}(t)|$ now follows precisely as in § 3.

REFERENCES

[1] J. O. C. EZEILO (1977) – « Atti Accad. Naz. Lincei, Rend. Cl. Sci. fis. Mat. e nat. », Ser. VIII, *63*, 212–224.

[2] R. REISSIG, G. SANSONE and R. CONTI (1974) – *Non-linear differential equations of higher order*, Noordhoff International Publishing, Leyden.