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James O.C. Ezeilo

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RENDICONTI

DELLE SEDUTE

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Classe di Scienze fisiche, matematiche e naturali

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Presiede il Presidente della Classe ANTONIO CARRELLI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Equazioni differenziali ordinarie. — *Extension of certain instability theorems for some fourth and fifth order differential equations.* Nota di JAMES O. C. EZEILO, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si estendono alcuni teoremi di instabilità alle soluzioni di alcune classi di equazioni differenziali ordinarie del quarto e quinto ordine.

1. The instability results of primary interest here are the (sole) instability theorem which was established in [1] for the fourth order differential equation:

$$x^{(4)} + a_1 \dddot{x} + a_2 \ddot{x} + a_3 \dot{x} + f(x) = 0 \qquad (f(0) = 0), \tag{1.1}$$

and an analogous one established as [2; Theorem 1] for the fifth order differential equation:

$$x^{(5)} + a_1 x^{(4)} + a_2 \dddot{x} + a_3 \ddot{x} + a_4 \dot{x} + f(x) = 0 \qquad (f(0) = 0). \tag{1.2}$$

Here $a_1, a_2, \cdots, a_5$ are constants and f was assumed in [1] and [2] to have a continuous derivative $f'(x)$, although the existence of $f'(x)$ was eventually shown in [2; Theorem 2] to be unnecessary for (1.2) in the two cases: (*i*) $a_1 > 0$ and $a_3 \leq 0$, (ii) $a_1 < 0$ and $a_3 \geq 0$.

We shall see here that the existence and continuity of $f'(x)$ can indeed be dispensed with altogether for (1.1) generally and for (1.2) if $a_1 \neq 0$ (regard-

(*) Nella seduta del 21 aprile 1979.

less of the sign of a_3), but the main object of the present note is to extend the aforementioned instability results to much more general equations in which the coefficients a_2, a_3 in (1.1) and the coefficients a_3, a_4 in (1.2) are not necessarily constants.

2. *Statement of results.* Our first result concerns the equations

$$x^{(4)} + a_1 \dddot{x} + h(x, \dot{x}, \ddot{x}, \dddot{x}) \ddot{x} + g(x) \dot{x} + f(x) = 0 \qquad (f(0) = 0), \tag{2.1}$$

where a_1 is a constant and f, g, h are continuous functions depending only on the arguments shown. We shall prove, in generalization of the theorem in [1] that

THEOREM 1. *If*

$$x_1^{-1} f(x_1) - \frac{1}{4} h^2(x_1, x_2, x_3, x_4) > 0 \qquad (x_1 \neq 0) \tag{2.2}$$

for arbitrary values of x_2, x_3 and x_4, then the trivial solution $x = 0$ of (2.1) *is unstable for all arbitrary a_1, g.*

Note here that, even when $h \equiv a_2$ (constant) and $f'(x)$ exists, our (2.2) here is decidedly weaker than the corresponding condition in [1] which requires that $\inf_x f'(x) > \frac{1}{4} a_2^2$.

We move next to the fifth order differential equation

$$x^{(5)} + a_1 x^{(4)} + a_2 \dddot{x} + \psi(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}) \ddot{x} + \phi(x) \dot{x} + f(x) = 0 \qquad (f(0) = 0), \tag{2.3}$$

where ψ, ϕ, f are continuous functions depending only on the arguments shown and a_1, a_2 are constants with $a_1 \neq 0$. We shall prove here that

THEOREM 2. *If*

$$x_1^{-1} f(x_1) \operatorname{sgn} a_1 - \frac{1}{4} \psi^2(x_1, x_2, x_3, x_4, x_5) \, | a_1 |^{-1} > 0 \qquad (x_1 \neq 0) \tag{2.4}$$

ior arbitrary values of x_2, x_3, x_4 and x_5, then the trivial solution $x = 0$ of (23)*. fs unstable for all arbitrary ϕ.*

Observe that when specialized to the case $\psi \equiv a_3$ (constant), (2.4) reduces in the case $a_1 > 0$ to:

$$x^{-1} f(x) > \frac{1}{4} a_3^2 a_1^{-1}$$

which is a significant improvement on the analogous restriction on f' in [2] where we had a lower bound of $a_3^2 a_1^{-1}$ on $f'(x)$ as the condition for instability when a_3 and a_1 are both positive.

3. *Proof of Theorem 1.* The procedure, as in [1] is by the Krasovskii technique, and our main tool is the function $V = V(x_1, x_2, x_3, x_4)$ given by

$$(3.1) \qquad V = x_2(x_3 + \tfrac{1}{2} a_1 x_2) - x_1(x_4 + a_1 x_3) - \int_0^{x_1} sg(s)\, ds .$$

Clearly

$$V(0, \varepsilon^2, \varepsilon, 0) = \varepsilon^3 + \tfrac{1}{2} a_1 \varepsilon^4 > 0$$

for all sufficiently small $\varepsilon > 0$. Thus V is capable of assuming positive values arbitrarily close to the origin in the (x_1, x_2, x_3, x_4)—space as one expects of a " Krasovskii " function.

Consider next the equivalent differential system

$$(3.2) \quad \dot{x}_1 = x_2 , \dot{x}_2 = x_3 , \dot{x}_3 = x_4 , \dot{x}_4 = - a_1 x_4 - h(x_1, x_2, x_3, x_4) x_3 - g(x_1) x_2 - f(x_1)$$

obtained by setting $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \ddot{x}$, $x_4 = \dddot{x}$ in (2.1); and let

$$(x_1, x_2, x_3, x_4) = (x_1(t), x_2(t), x_3(t), x_4(t))$$

be any solution of (3.2). A straightforward calculation from (3.1) and (3.2) will verify that

$$\begin{aligned} \dot{V} &\equiv \frac{d}{dt} V(x_1, x_2, x_3, x_4) \\ &= x_3^2 + x_1 x_3 h(x_1, x_2, x_3, x_4) + x_1 f(x_1) \\ &= [x_3 + \tfrac{1}{2} x_1 h(x_1, x_2, x_3, x_4)]^2 + \left\{ x_1 f(x_1) - \frac{1}{4} x_1^2 h^2(x_1, x_2, x_3, x_4) \right\} . \end{aligned}$$

Thus if $f(0) = 0$ and (2.2) holds then $\dot{V}$ is positive semi definite. Furthermore $\dot{V} = 0$ for all $t \geq 0$ necessarily implies that $x_1 = 0$ and therefore also that

$$x_2 = \dot{x}_1 = 0 \quad , \quad x_3 = \ddot{x}_1 = 0 \quad , \quad x_4 = \dddot{x}_1 = 0$$

for all $t \geq 0$. Thus, in spite of our $\dot{V}$ here not being positive definite as in [1], V nevertheless has all the essential ingredients in Krasovskii's criteria [3], and Theorem 1 is thereby established.

4. *Proof of Theorem 2.* Again our proof here will be by the Krasovskii technique, relevant details of which have been incidentally spelt out in [2; § 3].

Let $\Phi(x) = \int_0^x s\phi(s)\, ds$ and define the function $W = W(x_1, x_2, x_3, x_4, x_5)$ by

$$(4.1) \quad W = x_2(x_4 + a_1 x_3) - x_1(x_5 + a_1 x_4 + a_2 x_3) - \tfrac{1}{2} x_3^2 - \Phi(x_1) + \tfrac{1}{2} a_2 x_2^2 .$$

Consider now the function $V = V(x_1, x_2, x_3, x_4, x_5)$ defined by

$$V = W \operatorname{sgn} a_1 , \tag{4.2}$$

W being as defined in (4.1) and $a_1 \neq 0$. It is evident from (4.1) and (4.2) that

$$V(0, \varepsilon \operatorname{sgn} a_1, 0, \varepsilon^2, 0) = \varepsilon^3 + \tfrac{1}{2} a_2 \varepsilon^4 \operatorname{sgn} a ,$$
$$> 0$$

for all sufficiently small $\varepsilon > 0$. Thus V satisfies the Krasovskii property (P_1) of [2; § 3].

We turn next to the equivalent differential system for (2.3):

$$\begin{aligned} &\dot{x}_1 = x_2 \quad , \quad \dot{x}_2 = x_3 \quad , \quad \dot{x}_3 = x_4 \quad , \quad \dot{x}_4 = x_5 \\ &\dot{x}_5 = -a_1 x_5 - a_2 x_4 - \psi(x_1, x_2, x_3, x_4, x_5) x_3 - \phi(x_1) x_2 - f(x_1) . \end{aligned} \tag{4.3}$$

Let

$$(x_1, x_2, x_3, x_4, x_5) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t))$$

be any solution of (4.3). Then we have from (4.1), (4.2) and (4.3) that

$$\begin{aligned} \dot{V} &\equiv \frac{d}{dt} V(x_1, x_2, x_3, x_4, x_5) \\ &= |a_1| x_3^2 + \psi(x_1, x_2, x_3, x_4, x_5) x_1 x_3 \operatorname{sgn} a_1 + x_1 f(x_1) \operatorname{sgn} a_1 \\ &\equiv |a_1| \{x_3 + \tfrac{1}{2} x_1 \psi(x_1, x_2, x_3, x_4, x_5) |a_1|^{-1} \operatorname{sgn} a_1\}^2 + \\ &+ [x_1 f(x_1) \operatorname{sgn} a_1 - \frac{1}{4} \psi^2(x_1, x_2, x_3, x_4, x_5) x_1^2 |a_1|^{-1}] . \end{aligned}$$

Assume now that (2.4) holds. Then the term inside the square bracket is non negative so that $\dot{V}$ is clearly positive semi-definite, which is Krasovskii's property (P_2) of [2; § 3]. Furthermore $\dot{V} = 0$ for all $t \geq 0$ can only hold if

$$x_1(t) = 0 \qquad \text{for all} \quad t \geq 0$$

which in turn implies that

$$x_2 = \dot{x}_1 = 0 \quad , \quad x_3 = \ddot{x}_1 = 0 \quad , \quad x_4 = \dddot{x}_1 = 0 \quad , \quad x_5 = x_1^{(4)} = 0$$

for all $t \geq 0$; so that the remaining Krasovskii's property (P_3) of [2; § 3] also holds subject to (2.4). Theorem 2 now follows.

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