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Abstract monotone mappings and applications to functional differential equations

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Equazioni funzionali. — *Abstract monotone mappings and applications to functional differential equations.* Nota di MIHAI TURINICI, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — In questa Nota si dimostra un teorema di punto fisso per una classe di applicazioni monotone in senso astratto.

1. Throughout this Note, for every nonempty set X, $\mathcal{P}(X)$ denotes the class of all nonempty $Y \subset Y$. Let X be a nonempty set and let $\leq \subset X^2 = X \times X$ be an ordering on X. For every $x \in X$, $Y \subset X$, put $Y(x, \leq) = \{y \in Y ; x \leq y\}$. Denote by $\geq$ the associated dual ordering (i.e., $x \geq y$ iff $y \leq x$) [2, p. 3], and let $\supset\leq$ (resp. $\supset\geq$) denote the set-ordering defined on $\mathcal{P}(X)$ by: $Y \supset\leq Z$ (resp. $Y \supset\geq Z$) iff $Y \supset Z$ and, for every $y \in Y$ there is a $z \in Z$ with $y \leq z$ (resp. $y \geq z$).

Let (X, τ) be a topological space and let $\leq$ be an ordering on X. $\leq$ is said to be a closed ordering iff $X(x, \leq)$ is closed, for every $x \in X$. (X, τ) is said to be $\supset\leq$ (resp. $\supset\geq$)—compact iff, for every $\supset\leq$ (resp. $\supset\geq$)—directed family $\mathcal{U} \subset \mathcal{P}(X)$ of closed sets, $\cap \mathcal{U} \neq \varnothing$. Clearly, if (X, τ) is compact in the ordinary sense [5, ch. 5], it is also $\supset\leq$ (resp. $\supset\geq$)—compact, but the converse is not in general true (take $X = R_-$ (resp. R_+) with the usual topology and the usual (resp. dual) ordering).

2. Let X be a nonempty set, $\leq$ an ordering on X and $T : X \to X$ a monotone mapping (i.e., $x \leq y \Rightarrow Tx \leq Ty$) from X into itself. An useful result concerning the fixed points of T (a result that may be considered as a partial refinement of [1, Theorem 3]) may be stated as follows.

THEOREM 2.1. *Let* $X, \leq, T$ and $x \in X$ *satisfy*

(2.1) *every chain* $C \subset T(X)$ *has a supremum (infimum)*

(2.2) $x \leq Tx$ *(resp.* $x \geq Tx$*).*

Then, there exists a $v \in X$ *(resp. a* $u \in X$*) such that* (a) $v = Tv$ *(resp.* $u = Tu$*),* (b) $x \leq Tx$ *(resp.* $x \geq Tx$*)* $\Rightarrow x \leq v$ *(resp.* $x \geq u$*).*

Proof. Suppose $x \leq Tx$ (the proof for the dual ordering $\geq$ is similar) and put $Y = \{x \in X ; x \leq Tx\}$. From Hausdorff maximal principle [5, p. 33] there exists a maximal chain $L \subset Y$; with $x \in L$; as T is monotone, $T(L) \subset T(Y) \subset T(X)$ is also a chain in $T(X)$ and so, from (2.1), there exists

(*) Nella seduta del 10 marzo 1979.

$v = \sup T(L)$. Since $x \leq Tx \in T(L)$, we get $x \leq v$, which gives (again by the monotonicity of T) $Tx \leq Tv, \forall x \in L$, and therefore (from the definition of the supremum) $v \leq Tv$, i.e., $v \in Y$. Now, since $x \leq v, \forall x \in L$, we infer (from the maximality of L) $v \in L$. As $v = \sup T(L)$ and $Tv \in T(L)$, we get $Tv \leq v$ and thus (combining with the preceding relation) $v = Tv$, completing the proof. Q.E.D.

Remark 2.1. A sufficient condition for (2.1) is

(2.3) every chain $C \subset X$ has a supremum (infimum).

COROLLARY 2.1. *Suppose that, in the above theorem, conditions* (2.1) *and* (2.2) *are replaced respectively, by*

(2.1)′ *every chain* $C \subset T(X)$ *has in the same time a supremum and an infimum*

(2.2)′ *either* $x \leq Tx$ *or* $x \geq Tx$,

Then, there exist $v, u \in X$ *such that* (a) $v = Tv, u = Tu$, (b) $x \leq Tx$ (*resp.* $x \geq Tx) \Rightarrow x \leq v$ (*resp.* $x \geq u$), (c) $x = Tx \Rightarrow u \leq x \geq v$.

Remark 2.2. Under the conditions of the above corollary, it is justified to call v (resp. u) a maximal (resp. minimal) solution of the equation $x = Tx$.

3. Let (X, τ) be a topological space, $\leq$ an ordering on X and T a monotone mapping from X into itself. The main result of this Note (a result that may be considered as a "topological" version of Theorem 2.1) is the following.

THEOREM 3.1. *Let* $(X, \tau), \leq, T$ and $x \in X$ *satisfy*

(3.1) *both* $\leq$ *and* $\geq$ *are closed orderings*

(3.2) $\overline{T(X)}$ (*the closure of* $T(X)$) *is both* $\supset\leq$ *and* $\supset\geq$*—compact*

(3.3) *either* $x \leq Tx$ *or* $x \geq Tx$.

Then, there exist $v, u \in X$ *such that* (a) $v = Tv, u = Tu$, (*b*) $x \leq Tx$ (*resp.* $x \geq Tx) \Rightarrow x \leq v$ (*resp.* $x \geq u$), (c) $x = Tx \Rightarrow u \leq x \leq v$.

Proof. Let $C \subset T(X)$ be an arbitrary chain in $T(X)$. Firstly, we prove that for every $z \in \overline{C}, x \in C$, z and x are comparable. Indeed, suppose z and x are not comparable. From (3.1) there exists a neighborhood $U \in \mathcal{V}(z)$ such that, for every $u \in U$, u and x are not comparable. Let $V \in \mathcal{V}(z)$ be arbitrary (without loss of generality we may suppose $V \subset U$). As our assumption implies $z \in \overline{C} \setminus C$, there exists a $y \in C$ such that $y \in V \subset U$, i.e., y and x are not comparable, a contradiction, proving our assertion. In this case, the family $\{\overline{C}(x, \leq); x \in C\} \subset \mathcal{P}(\overline{T(X)})$ is a $\supset\leq$—directed family of closed sets and so, from (3.2), $\cap \{\overline{C}(x, \leq); x \in C\} = M$ is nonempty and closed. Furthermore, for every $w \in M$, $C \subset X(w, \geq)$, and this gives $\overline{C} \subset \overline{X(w, \geq)} = X(w, \geq)$. From this fact, M will consist of a single element, i.e., $M = \{z\}$, for some $z \in X$. Firstly, from the above remark, $x \leq z$, $\forall x \in C$. Now, let $u \in X$ be such that $x \leq u$, $\forall x \in C$. This means that $C \subset X(u, \geq)$ and thus $z \in \overline{C} \subset \overline{X(u, \geq)} = X(u, \geq)$, i.e., $z \leq u$, showing that $z = \sup C$. An analogous reasoning may be done for the dual ordering $\geq$, and so, (2.1)′ holds. On the other hand, (3.3) coincides with (2.2)′. Therefore, corollary 2.1 applies, and this completes the proof. Q.E.D.

Let A be a nonempty set and let $\leqq$ be an ordering on A. A family $\{T_a ; a \in A\}$ of mappings from X into itself is said to be a monotone family iff, for every $a, b \in A$, $a \leqq b$, and $x \in X$, we have $T_a x \leq T_b x$.

Now, suppose $\{T_a ; a \in A\}$ is a monotone family of monotone mappings from X into itself, having a unique maximal and minimal fixed point.

COROLLARY 3.1. *Suppose that, for every* $a \in A$, *conditions of Theorem* 3.1 *are satisfied, with* T *replaced by* T_a. *Furthermore, for every* $a \in A$, *let* $S(a)$ (*resp.* $s(a)$) *denote the maximal* (*resp. minimal*) *solution of the equation* $x = T_a x$. *Then, necessarily, the mappings* $S : A \to X$ *and* $s : A \to X$ *are monotone.*

4. Let $n \geq 1$ be a positive integer and let $(R^n, \|\cdot\|)$ be the euclidean n-dimensional space endowed with a norm $\|\cdot\|$. Furthermore let $\{I, J\}$ be a partition of $\{1, \cdots, n\}$. Define an ordering $\leq$ on R^n by

i) $\quad (x_1, \cdots, x_n) \leq (y_1, \cdots, y_n)$ iff $x_i \leq y_i, \quad \forall i \in I, x_j \geq y_j, \quad \forall j \in J$

where, in the right hand, $\leq$ denotes the usual ordering on R, and $\geq$ its dual. Clearly, $\leq$ is a closed ordering on R^n.

In what follows, X (resp. A) denotes the set of all continuous $x : R_+ \to R^n$ (resp. $a : R_+ \to R_+$). For every $x \in X$, define $\|x\| \in A$ by

ii) $\quad \|x\|(t) = \|x(t)\|, \quad \forall t \in R_+$

and $|x| \in A$ by

iii) $\quad |x|(t) = \sup \{\|x(s)\| ; s \in [0, t]\}, \quad \forall t \in R_+$.

It is well known that X is a locally convex space, with the topology defined by the directed family of seminorms $\mathscr{S} = \{|\cdot|(t) ; t \in R_+\}$ Denote also by $\leq$ the ordering on X induced by the ordering $\leq$ on R^n, in the usual way, i.e.,

iv) $\quad x \leq y$ iff $x(t) \leq y(t), \quad \forall t \in R_+$.

Clearly, $\leq$ is a closed ordering on X and so is $\geq$ its dual.

Let $t \mapsto \hat{t} \in \mathscr{P}(R_+)$ be a given mapping. Denote for simplicity $\hat{R}_+ = \{\hat{t} ; t \in R_+\}$. Let $x^0 \in R^n$ and $k : X \times \hat{R}_+ \to R^n$. Then, we may consider (formally) the functional differential equation

$$x'(t) = k(x, \hat{t}), \qquad \forall t \in R_+ ; \qquad x(0) = x^0 \tag{4.1}$$

and the associated functional integral inequalities

$$x(t) \leq x^0 + \int_0^t k(x, \hat{s})\, ds, \qquad \forall t \in R_+ \tag{4.2}$$

$$x(t) \geq x^0 + \int_0^t k(x, \hat{s})\, ds, \qquad \forall t \in R_+ . \tag{4.3}$$

The main result concerning (4.1)–(4.3) may be stated as follows.

THEOREM 4.1. *Suppose there exist* $g \in A$, $K : A \times \hat{R}_+ \to R_+$, *and* $x \in X$ *such that (denoting* $X_1 = \{x \in X ; \|x\| \leq g\}$ *and* $A_1 = \{a \in A ; a \leq g\}$)

(4.4) $\forall x \in X_1$ *the map* $t \mapsto k(x, \hat{t})$ *is continuous*

(4.5) $\forall a \in A_1$ *the map* $t \mapsto K(a, \hat{t})$ *is continuous*

(4.6) k *is monotone* $(x, y \in X_1, x \leq y \Rightarrow k(x, \hat{\cdot}) \leq k(y, \hat{\cdot}))$

(4.7) K *is monotone* $(a, b \in A_1, a \leq b \Rightarrow K(a, \hat{\cdot}) \leq K(b, \hat{\cdot}))$

(4.8) $\|k(x, \hat{\cdot})\| \leq K(\|x\|, \hat{\cdot}), \qquad \forall x \in X_1$

(4.9) $\|x^0\| + \int_0^t K(g, \hat{s})\, ds \leq g(t), \qquad \forall t \in R_+$

(4.10) $x \in X_1$ *and satisfies at least one of the associated functional integral inequalities* (4.2), (4.3).

Then, there exist $v, u \in X_1$, *such that* (a) v *and* u *are solutions of* (4.1), (b) *if* $x \in X_1$ *is a solution of* (4.2) (*resp.* (4.3)) *then* $x \leq v$ (*resp.* $x \geq u$), (c) *if* $x \in X_1$ *is a solution of* (4.1), *then* $u \leq x \leq v$.

Proof. Let $T : X_1 \to X$ be defined, for every $x \in X_1$, by

$$Tx(t) = x^0 + \int_0^t k(x, \hat{s})\, ds, \qquad \forall t \in R_+ . \tag{4.11}$$

From (4.8)+(4.9), $T(X_1) \subset X_1$, i.e., $\|Tx\| \leq g$, $\forall x \in X_1$. On the other hand, from (4.7)+(4.8), $\|(Tx)'\| = \|k(x, \hat{\cdot})\| \leq K(\|x\|, \hat{\cdot}) \leq K(g, \hat{\cdot})$, $\forall x \in X_1$. So from the well known Arzelà–Ascoli theorem [4], [5, p. 234], $T(X_1)$ is relatively compact. Finally, (4.6) says that T is monotone. Thus, Theorem 3.1 is entirely applicable, and this completes the proof. Q.E.D.

Remark 4.1. As in § 2, it is justified to call v (resp. u) a maximal (resp. minimal) solution of (4.1).

Remark 4.2. In the above theorem, the mapping T defined by (4.11) is not in general continuous. Therefore, Banach's fixed point theorem, as well as Schauder–Tychonoff's fixed point theorem (see, e.g., [4]) are not in general applicable.

Remark 4.3. An useful application of these methods might be done in the context of projective functional (differential) equations, [3], [6], in which the class of (abstract) monotone mappings plays an essential role.

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