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On Groups Having Exactly 2 Conjugacy Classes of Maximal Subgroups

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 10 marzo 1979

Presiede il Presidente della Classe ANTONIO CARRELLI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Algebra. — *On Groups Having Exactly 2 Conjugacy Classes of Maximal Subgroups*. Nota di SAAD ADNAN, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — Si apportano alcuni contributi diretti a provare una congettura relativa ai gruppi finiti dotati di due sole classi di coniugio di sottogruppi massimali.

CONJECTURE. If the finite group G has exactly 2 conjugacy classes of maximal subgroups, then $G = PQ$ where P and Q are S_p and S_q subgroups of G, $P \lhd G$ and Q is cyclic. Further, Q acts irreducibly on $P/\phi(P)$.

INTRODUCTION. Among the known finite simple groups there is no group which has exactly 2 conjugacy classes of maximal subgroups. As we shall see later, the main difficulty of the conjecture is to prove non-simplicity. The simple group of order 168 comes nearly as a counterexample to the conjecture since it has exactly 3 conjugacy classes of maximal subgroups two of which are interchanged by an outer automorphism of order 2.

We have not been able to prove the conjecture as it stands. However, if the maximal subgroups behave nicely, we have the following main results:

THEOREM A. *Let* G *be a finite group having exactly 2 conjugacy classes of maximal subgroups. If all the maximal subgroups of G are Hall subgroups, then* $G = PQ$ *where* P *is an elementary abelian normal* S_p*-subgroup of* G *and* Q *is a cyclic* S_q*-subgroup of* G *of prime order. Further,* Q *acts irreducibly on* P.

(*) Nella seduta del 13 gennaio 1979.

THEOREM B. *Let* G *be a finite group in which every maximal subgroup has index a power of a prime. If* G *has exactly* 2 *conjugacy classes of maximal subgroups, then* $G = PQ$ *where* P *and* Q *are* S_p*-and* S_q*-subgroups of* G, $P \lhd G$ *and* Q *is cyclic. Further,* Q *acts irreducibly on* $P/\phi(P)$.

Symbols and Notations. The symbols and notations conform to [2].

LEMMA 1. *Let* G *be a finite group and let* H *be a subgroup of* G. *If* P *is an* S_p*-subgroup of* G *such that* $P \subseteq H$ *and* $N_G(P) \subseteq g^{-1}Hg$, *some* $g \in G$, *then* $g \in H$.

Proof. We have $P, P^g \subseteq H^g$. By Sylow's theorems, there is an element $x \in H^g$ such that $P^x = P^g$ i.e. $gx^{-1} \in N_G(P) \subseteq H^g$ i.e. $g \in H^g$ and so $g \in H$.

LEMMA 2. *Let* G *be a finite group with exactly* 2 *conjugacy classes of maximal subgroups. If* M *and* N *are non-conjugate maximal subgroups of* G, *then* $G = MN$.

Proof. Let $p \in \pi(G)$ and let P be an S_p-subgroup of G. Let L be a maximal subgroup of G containing P. By hypothesis $L = N^g$ or $L = M^g$, some $g \in G$. Thus MN contains an S_p-subgroup of G for each $p \in \pi(G)$. The lemma follows.

LEMMA 3. *Let* G *be a finite group having exactly* 2 *conjugacy classes of maximal subgroups. If* G *is not simple then* G *has a non-trivial nilpotent normal subgroup.*

Proof. Let M and N, be non-conjugate maximal subgroups of G and let H be a non-trivial normal subgroup of G. Clearly we may assume that $H \subseteq M$. If $H \subseteq N$, then $H \subseteq \phi(G)$ and we are done. Hence, we may assume $1 \neq [H : H \cap N]$. Let p be a prime divisor of $[H : H \cap N]$ and P be an S_p-subgroup of H. Since $G = HN_G(P)$, $N_G(P)$ does not lie in any conjugate of M. Also, since $|H \cap N| = |H \cap N^g|$, $N_G(P)$ does not lie in any conjugate of N. We conclude that $N_G(P) = G$ and so the lemma follows.

LEMMA 4. *Let* G *be a finite soluble group having exactly* 2 *conjugacy classes of maximal subgroups. Then* $G = PQ$ *where* P *and* Q *are* S_p*–and* S_q*-subgroups of* G, $P \lhd G$, Q *is cyclic. Further,* Q *acts irreducibly on* $P/\phi(P)$.

Proof. (*i*) $G = PQ$, $P \lhd G$ and Q is cyclic: Since G has exactly 2 conjugacy classes of maximal subgroups, the solubility of G implies $G = PQ$. Let L be a normal subgroup of G of prime index, q say. It is claimed that $P \lhd G$. For if P is an S_p-subgroup of G contained in L, then $G = LN(P)$. If $P \ntriangleleft G$, then choose maximal subgroups U, V of G such that $N_G(P) \subseteq U$, $Q \subseteq V$. Clearly no two of U, V and L are conjugate in G contrary to hypothesis. Thus $P \lhd G$.

Now let T_1, T_2 be maximal subgroups of Q. Since G possesses exactly 2 conjugacy classes of maximal subgroups and $PT_i \lhd G$, $i = 1, 2$, we must have $M = PT_1 = PT_2$ and T_i is an S_q-subgroup of M. However, $T_1 \subseteq N_M(T_2)$ and we conclude that $T_1 = T_2$ i.e. Q has a unique maximal subgroup and so Q is cyclic.

(*ii*) Q acts irreducibly on $P/\phi(P)$: We proceed by induction on $|G|$. If $1 \neq \phi(P)$, then the assertion holds for $G/\phi(P)$, that is $Q\phi(P)/\phi(P) \simeq Q$ acts irreducibly on $P/\phi(P)$.

Therefore we may assume that P is elementary abelian. By Maschke's theorem ([2], p. 66), $P = \prod_{i=1}^{j} M_i$ where M_i is a minimal normal subgroup of G. But then for $1 \leq i_0, j_0 \leq j$, $Q \prod_{i \neq i_0} M_i$, $Q \prod_{i \neq j_0} M_i$ are maximal subgroups of G which are conjugate in G if and only if $i_0 = j_0$. If T is the unique maximal subgroup of Q, then PT is the unique maximal subgroup of G containing P. Thus G has exactly $j + 1$ conjugacy classes of maximal subgroups forcing $j = 1$ and proving the lemma.

Remark. Before proceeding to lemma 5, we note that lemmas 3 and 4 show that a minimal counterexample G to the conjecture stated at the beginning of the present paper is simple.

LEMMA 5. *Let* G *be a finite simple group possessing exactly* 2 *conjugacy classes of maximal subgroups. Let* M *and* N *be non-conjugate maximal subgroups of* G. *If p is a prime such that* $p \in \pi(M) - \pi(N)$, *then* M *is p-strongly embedded in* G.

Proof. Let P be an S_p-subgroup of G and let M be a maximal subgroup of G containing $N_G(P)$. Choose $g \in G - M$ such that $P \cap M^g = P_0$ is of maximal order. By lemma 1, $P_0 < P$. It is claimed that $P_0 = 1$. For suppose by way of contradiction that $P_0 \neq 1$. Then $N_G(P_0) \subseteq M^x$, some $x \in G$. If $x \notin M$, then $P \cap M^x = P \cap N_G(P_0) > P_0$ a contradiction to the maximality of P_0. On the other hand if $x \in M$, then $P \cap P_1 = P_0$ for some S_p-subgroup P_1 of M^g. Thus $P_1 \cap N_G(P_0) \subseteq P_2$ for some S_p-subgroup P_2 of M. By Sylow's theorems $P_2 = P^m$, for some $m \in M$. Therefore $|P \cap M^{gm^{-1}}| = |P^m \cap M^g| \geq$ $\geq |N_{P_1}(P_0)| > |P_0|$. By maximality of P_0 we conclude that $gm^{-1} \in M$ i.e. $g \in M$ a final contradiction. Lemma 5 is proved.

We are now in a position to prove both theorems A and B. We start by theorem B.

Proof of Theorem B. Let G be a minimal counterexample to the theorem. By lemma 5 of [1], G is not simple. By lemma 3 above, G has a non-trivial nilpotent normal subgroup H say. By minimality of G, G/H is soluble. Since H is nilpotent, G is soluble. The conclusion now follows from lemma 4 above.

Proof of Theorem A. We introduce the following sets of primes:

$$\mu = \{p \mid p \in \pi(M)\}, \lambda = \{p \mid p \in \pi(N)\}$$

and

$$\nu = \mu \cap \lambda. \quad \text{Clearly} \quad \nu = \pi(M \cap N).$$

Now let G be a minimal counterexample to the theorem.

(*i*) G is not simple: Since M is a Hall subgroup of G we have $|M| = \prod_{i=1}^{l} p_i^{\alpha_i} \prod_{i=1}^{m} q_i^{\beta_i}$ where $p_i \in \mu - \nu$, $q_i \in \nu$ and $p_i^{\alpha_i}$ is the order of an S_{p_i}-subgroup of G. Similarly $|N| = \prod_{i=1}^{n} r_i^{\gamma_i} \prod_{i=1}^{m} q_i^{\beta_i}$ where $r_i \in \lambda - \nu$.

If G is not simple, then by lemma 5, p_i does not divide $|M \cap M^g|$ for all $g \in G - M$. Hence for $g \in G - M$, we have:

$$|MM^g| = \frac{|M|^2}{|M \cap M^g|} \geq (\Pi\, p_i^{\alpha_i})^2\, \Pi\, q_i^{\beta_i} \text{ and for } g \in G - N \text{ we have:}$$

$$|NN^g| = \frac{|N|^2}{|N \cap N^g|} \geq (\Pi\, r_i^{\gamma_i})^2\, \Pi\, q_i^{\beta_i}.$$

Since by lemma 2, $G = MN$, we have:

$|G| = |MN| = \dfrac{|M|\,|N|}{|M \cap N|} = \Pi\, p_i^{\alpha_i}\, \Pi\, q_i^{\beta_i}\, \Pi\, r_i^{\gamma_i}$. It is clear now that $|MN^g| >$ $> |G|$ or $|NN^g| > |G|$, a contradiction. We conclude that G is not simple.

(*ii*) Theorem A holds: By lemma 3 and the minimality of G, G is soluble. By lemma 4, $G = PQ$, $P \Delta G$ and Q is cyclic. Since a maximal subgroup of G containing $Q\phi(P)$ cannot be a Hall subgroup of G, $\phi(P) = 1$ and so P is elementary abelian and Q acts irreducibly on P. Finally since a maximal subgroup of G containing PT, T subgroup of Q, cannot be a Hall subgroup of G, Q must be cyclic of prime order.

References

[1] S. Adnan (1976) - *A Characterisation of PSL* (2, 7), « J. London Math. Soc. » (2) *12*.
[2] D. Gorenstein (1969) - *Finite Groups* (Harper and Row).