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On the existence of periodic solutions of certain third order non-dissipative differential systems

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Equazioni differenziali ordinarie. — *On the existence of periodic solutions of certain third order non-dissipative differential systems.* Nota di JAMES O. C. EZEILO, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si dànno condizioni sufficienti perché l'equazione $\dddot{X} + A\ddot{X} + B\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X})$, con A, B, C matrici $n \times n$ simmetriche con elementi costanti $P(t + \omega, X, Y, Z) = P(t, X, Y, Z)$ ammetta almeno una soluzione periodica.

1. We shall be concerned here with real third order differential systems of the form:

$$\dddot{X} + A\ddot{X} + B\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}) \tag{1.1}$$

in which A, B are constant symmetric $n \times n$ matrices and X, H, P are n-vectors with H, P dependent only on the arguments shown. It will be assumed as basic throughout that $P(t, X, Y, Z)$ is continuous and ω-periodic in t (that is $P(t + \omega, X, Y, Z) = P(t, X, Y, Z)$ for some $\omega > 0$), and that there are constants $\delta \geq 0$, $\varepsilon \geq 0$ such that

$$\| P(t, X, Y, Z) \| \leq \delta + \varepsilon (\| X \| + \| Y \| + \| Z \|) \tag{1.2}$$

for arbitrary t, X, Y and Z. Here and elsewhere the symbol $\| \cdot \|$ denotes the Euclidean norm. The derivatives $\partial h_i / \partial x_j$, h_i and x_j $(1 \leq i, j \leq n)$ here and elsewhere being the components of H and X respectively, are also assumed continuous, with the (Jacobian) matrix $J_h(X) \equiv (\partial h_i / \partial x_j)$ symmetric, for arbitrary X.

Let $\lambda_i(A)$, $\lambda_i(B)$, $\lambda_i(J_h(X))$ denote the eigenvalues (all real) of A, B, J_h respectively and let $\alpha \equiv \max_{1 \leq i \leq n} \lambda_i(A)$, $\beta = \max_{1 \leq i \leq n} \lambda_i(B)$. The following result, extending an earlier (scalar) result in [1], was announced, but without proof, at the International Congress Mathematicians in Helsinki (in August 1978):

THEOREM. *There exists a constant* $\varepsilon_0 = \varepsilon_0(\delta, A, B, H) > 0$ *such that if*

$$\gamma_0 \equiv \inf_{i,X} \lambda_i(J_h(X)) > \begin{cases} 0, & \text{if one at least of } \alpha, \beta \text{ is non positive}, \\ \alpha\beta, & \text{if } \alpha \text{ and } \beta \text{ are both positive}, \end{cases} \tag{1.3}$$

then (1.1) *has at least one* ω*-periodic solution provided that* $\varepsilon \leq \varepsilon_0$.

(*) Nella seduta del 10 febbraio 1979.

The object of the present note is to supply now a detailed proof of this theorem.

The reference here, in the title, to the system (1.1) as non-dissipative stems from the condition (1.3) which is clearly " non Routh-Hurwitz ". Note that there is no loss in generality in assuming that $H(0) = 0$; for the subtraction of $H(0)$ from either side of (1.1) gives an equation with H, P replaced by H_0, P_0, where $H_0(X) = H(X) - H(0)$ which satisfies $H_0(0) = 0$ and $P_0 = P - H(0)$ which satisfies

$$\| P_0(t, X, Y, Z) \| \leq \delta_0 + \varepsilon (\| X \| + \| Y \| + \| Z \|)$$

with $\delta_0 \equiv \delta + \| H(0) \|$, which is the same as (1.2).

2. NOTATION. In what follows we shall use γ's with or without suffixes to denote positive constants whose magnitudes depend only on δ, A, B and H. The γ's without suffixes are not necessarily the same in each place of occurence but the numbered γ's : $\gamma_0, \gamma_1, \gamma_2, \cdots$ retain a fixed magnitude throughout.

Next, given any pair of vectors, X and Y say, with components $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ respectively, we shall use $\langle X, Y \rangle$ to denote their scalar product $\sum_{i=1}^{n} x_i y_i$. Thus, in particular $\langle X, X \rangle \equiv \| X \|^2$.

3. The proof is by the Leray-Schauder technique, with (1.1) embedded in the parameter-dependent equation:

$$\dddot{X} + \mu (A\ddot{X} + B\dot{X}) + (1 - \mu)\gamma_0 X + \mu H = \mu P \tag{3.1}$$

where the parameter μ is as usual restricted to the closed range $[0, 1]$. Note that, when $\mu = 0$, (3.1) reduces to the equation

$$\dddot{X} + \gamma_0 X = 0$$

which clearly has no non-trivial ω-periodic solution. Also, when $\mu = 1$, (3.1) reduces to the equation (1.1). Thus the theorem will follow from the usual Leray-Schauder fixed point considerations (see for example theorem 1.39 of [3]) if it can be shown that there are constants $\gamma_1, \gamma_2, \gamma_3$ all independent of μ, such that

$$\| X \| \leq \gamma_1, \| \dot{X} \| \leq \gamma_2 \quad \text{and} \quad \| \ddot{X} \| \leq \gamma_3 \qquad (0 \leq t \leq \omega) \tag{3.2}$$

for every ω-periodic solution of (3.1) corresponding to $0 \leq \mu \leq 1$.

4. *Preliminary lemmas.* We shall make occasional use of the following lemmas:

LEMMA 1. *Let* D *be a symmetric* $n \times n$ *matrix and* X *any n-vector. Then*

$$d_1 \| X \|^2 \leq \langle X, DX \rangle \leq d_2 \| X \|^2 \tag{4.1}$$

where d_1, d_2 *are respectively the least and the greatest of the eigenvalues of* D.

This is a well known result (see for example [2; p. 288]).

LEMMA 2. *If* $X = X(t)$ *is twice continuously differentiable in* t, *then*

$$\int \langle \ddot{X}, H(X) \rangle \, dt = \langle \dot{X}, H(X) \rangle - \int \langle J_h(X) \dot{X}, \dot{X} \rangle \, dt, \tag{4.2}$$

the integrals here being indefinite integrals.

Proof. Since $\langle \ddot{X}, H(X) \rangle \equiv \sum_{i=1}^{n} \ddot{x}_i h_i$ we have on integrating by parts, that

$$\int \langle \ddot{X}, H(X) \rangle \, dt = \sum_{i=1}^{n} \dot{x}_i h_i - \int \sum_{i=1}^{n} \dot{x}_i \frac{dh_i}{dt} \, dt$$

$$= \langle \dot{X}, H \rangle - \int \sum_{i=1}^{n} \sum_{k=1}^{n} \dot{x}_i \frac{\partial h_i}{\partial x_k} \dot{x}_k \, dt$$

$$= \langle \dot{X}, H \rangle - \int \langle J_h(X) \dot{X} \ \dot{X} \rangle \, dt$$

which establishes (4.2).

Throughout what follows $X = X(t)$ denotes an arbitrary ω-periodic solution of (3.1) with μ restricted always to be the range $0 \leq \mu \leq 1$. The objective now will be to establish (3.2).

The main tool is the scalar function $u = u(t)$ given by

$$u = \tfrac{1}{2} b_1 \langle \ddot{X}, \ddot{X} \rangle - b_2 \langle X, \ddot{X} \rangle + \langle \dot{X}, \ddot{X} \rangle \tag{5.1}$$

where $b_1 > 0$, $b_2 > 0$ are constants whose values are as yet undetermined but will be fixed to advantage as γ's in the course of the proof. We have, by an elementary differentiation with respect to t that

$$\dot{u} = - b_1 \langle \ddot{X}, \mu A \ddot{X} + \mu B \dot{X} + (1-\mu) \gamma_0 X + \mu H - \mu P \rangle - b_2 \langle \dot{X}, \ddot{X} \rangle +$$
$$+ b_2 \langle X, \mu A \ddot{X} + \mu B \dot{X} + (1-\mu) \gamma_0 X + \mu H - \mu P \rangle + \langle \ddot{X}, \ddot{X} \rangle -$$
$$- \langle \dot{X}, \mu A \ddot{X} + \mu B \dot{X} + (1-\mu) \gamma_0 X + \mu H - \mu P \rangle .$$

Note that the terms

$$\langle \ddot{X}, B\dot{X} \rangle, \langle X, B\dot{X} \rangle, \langle \dot{X}, A\ddot{X} \rangle$$

which occur on the right hand side of (5.2) are perfect t-differentials since A, B, are symmetric. Also, since $J_h(X)$ is symmetric we have from equation 2.4 (3) of [2] that

$$\langle \dot{X}, H(X) \rangle = \frac{d}{dt} \int_0^1 \langle H(\sigma X), X \rangle \, d\sigma$$

so that the term $\langle \dot{X}, H(X) \rangle$ which occurs on the same right hand side of (5.2) is also a perfect t-differential. Thus we may indeed reset (5.2) in the form

$$\dot{u} \equiv u_1 + u_2 + u_3 \tag{5.3}$$

where

$$\begin{aligned} u_1 = & \{\langle \ddot{X}, \ddot{X} \rangle - \mu b_1 \langle \ddot{X}, A\ddot{X} \rangle\} + \\ & + \{ub_2 \langle X, A\ddot{X} \rangle - b_1 \langle \ddot{X}, (1-\mu)\gamma_0 X + \mu H \rangle - \mu \langle \dot{X}, B\dot{X} \rangle\} + \\ & + \{b_2 \langle X, (1-\mu)\gamma_0 X + \mu H \rangle\} \equiv \\ \equiv & \, u_{11} + u_{12} + u_{13}, \end{aligned}$$

say,

$$u_2 = -\mu \langle b_1 \ddot{X} + \dot{X} - b_2 X, P \rangle \tag{5.4}$$

and u_3 is a perfect t-differential. Hence, integrating both sides of (5.3) with respect to t from $t = 0$ to $t = \omega$, we have, X being ω-periodic, that

$$\int_0^\omega (u_{11} + u_{12} + u_{13})\, dt + \int_0^\omega u_2 \, dt = 0 . \tag{5.5}$$

Now, by Lemma I,

$$\langle \ddot{X}, A\ddot{X} \rangle \leq \alpha \| \ddot{X} \|^2$$

so that, since $0 \leq \mu \leq 1$,

$$u_{11} \geq (1 - \alpha b_1) \| \ddot{X} \|^2 . \tag{5.6}$$

Next, since $H(0) = 0$, we have from equation 2.2 (3) of [2] that $H(X) = \int_0^1 J_h(\sigma X) X d\sigma$ so that, in particular

$$\begin{aligned} \langle X, H(X) \rangle &= \int_0^1 \langle X, J_h(\sigma X) X \rangle \, d\sigma \\ &\geq \gamma_0 \| X \|^2 \end{aligned}$$

by (1.3) and (4.1); and hence

$$u_{13} \geq b_2 \gamma_0 \| X \|^2 \tag{5.7}$$

Finally, we have by Lemma 2 that

$$(5.8) \qquad \int_0^\omega \langle X , A\ddot{X}\rangle \, dt = -\int_0^\omega \langle A\dot{X} , \dot{X}\rangle \, dt$$

$$\geq -\alpha \int_0^\omega \| \dot{X} \|^2 \, dt$$

in view of (4.1), and then analagously for the terms $\int_0^\omega \langle \ddot{X} , X\rangle \, dt$ and $\int_0^\omega \langle \ddot{X} , H(X)\rangle \, dt$ appearing in $\int_0^\omega u_{12} \, dt$ that

$$(5.9) \qquad \int_0^\omega \langle \ddot{X} , X\rangle \, dt = -\int_0^\omega \| \dot{X} \|^2 \, dt$$

$$(5.10) \qquad \int_0^\omega \langle \ddot{X} , H(X)\rangle \, dt = -\int_0^\omega \langle J_h(X) \dot{X} , \dot{X}\rangle \, dt$$

$$\leq -\gamma_0 \int_0^\omega \| \dot{X} \|^2 \, dt ,$$

the latter inequality deriving immediately from the use of (1.3) and (4.1). Since

$$(5.11) \qquad \langle \dot{X} , B\dot{X}\rangle \leq \beta \| \dot{X} \|^2$$

it is clear from (5.8), (5.9), (5.10) and (5.11) that

$$(5.12) \qquad \int_0^\omega u_{12} \, dt \geq (b_1 \gamma_0 - b_2 \alpha - \beta) \int_0^\omega \| \dot{X} \|^2 \, dt$$

Thus we have from (5.6), (5.7) and (5.12) that

$$(5.13) \qquad \int_0^\omega (u_{11} - u_{12} + u_{13}) \, dt \geq \int_0^\omega u_4 \, dt$$

where

$$u_4 = (1 - \alpha b_1) \| \ddot{X} \|^2 + (b_1 \gamma_0 - b_2 \alpha - \beta) \| \dot{X} \|^2 + b_2 \gamma_0 \| X \| \|^2 .$$

A most crucial part of our proof is to show now that the so far undefined positive constants b_1, b_2 in (5.1) can in fact be fixed such that

$$u_4 \geq \gamma_4 (\| \ddot{X} \|^2 + \| \dot{X} \|^2 + \| X \|^2) \tag{5.14}$$

for some γ_4. We shall distinguish here two cases (already highlighted in (1.3)) namely: (I) *one at least of* α, β *is non positive*, (II) α *and* β *are both positive.*

We start with the case (I). Suppose for example that $\alpha \leq 0$. Then clearly $(1 - \alpha b_1) \geq 1$ for arbitrary $b_1 > 0$.

Also

$$b_1 \gamma_0 - b_2 \alpha - \beta \geq \gamma_5$$

if

$$b_1 \geq (\gamma_5 + |\beta|) \gamma_0^{-1}, \tag{5.15}$$

for arbitrary $b_2 > 0$. Thus when $\alpha \leq 0$ we have that

$$u_4 \geq (\| \ddot{X} \|^2 + \gamma_5 \| \dot{X} \|^2 + \gamma_0 \gamma_6 \| X \|^2)$$

if b_1 is fixed by (5.15) and $b_2 = \gamma_6$, which establishes (5.14) with $\gamma_4 = \min(1, \gamma_5, \gamma_0 \gamma_6)$. Suppose on the other hand that $\beta \leq 0$. Then, if $\alpha \leq 0$, $b_1 = \gamma_7 = b_2$ clearly secures the estimate:

$$u_4 \geq \| \ddot{X} \|^2 + \gamma_0 \gamma_7 (\| \dot{X} \|^2 + \| X \|^2)$$

which implies (5.14) (with $\gamma_4 = \min(1, \gamma_0 \gamma_7)$) while the choice

$$b_1 = \tfrac{1}{2} \alpha^{-1}, \; b_2 = \tfrac{1}{4} \gamma_0 \alpha^{-2}$$

when $\alpha > 0$ secures the estimate:

$$u_4 \geq \tfrac{1}{2} (\| \ddot{X} \|^2 + \tfrac{1}{2} \gamma_0 \alpha^{-1} \| \dot{X} \|^2 + \tfrac{1}{2} \gamma_0^2 \alpha^{-2} \| X \|^2)$$

which again implies (5.14) but with $\gamma_4 = \frac{1}{2} \min(1, \frac{1}{2} \gamma_0 \alpha^{-1}, \frac{1}{2} \gamma_0^2 \alpha^{-2})$. Thus whether $\alpha \leq 0$ or $\beta \leq 0$ it is possible to fix $b_1 = \gamma$, $b_2 = \gamma$ so that (5.14) holds.

We turn next to the case (II): $\alpha > 0$ and $\beta > 0$. Note that, since $\gamma_0 > \alpha\beta$, by (1.3), it is possible to choose γ_8 such that

$$\beta \gamma_0^{-1} < \gamma_8 < \alpha^{-1} \tag{5.16}$$

Now fix $b_1 = \gamma_8$ and $b_2 \equiv \frac{1}{2} \alpha^{-1} (\gamma_0 \gamma_8 - \beta) > 0$, by (5.16). Then

$$\begin{aligned} u_4 &\geq (1 - \alpha\gamma_8) \| \ddot{X} \|^2 + \tfrac{1}{2} (\gamma_8 \delta_0 - \beta) \| \dot{X} \|^2 + \tfrac{1}{2} \alpha^{-1} (\gamma_0 \gamma_8 - \beta) \| X \|^2 \\ &\geq \gamma (\| \ddot{X} \|^2 + \| \dot{X} \|^2 + \| X \|^2) \end{aligned}$$

for some γ, since $(1 - \alpha\gamma_8)$ and $(\gamma_8 \gamma_0 - \beta)$ are both positive, by (5.16). Thus in the case (II), (5.14) holds for some appropriate choice of b_1, b_2 as γ's. We have thus conclusively verified that, subject to (1.3), there exist γ_9, γ_{10} such that if

$$b_1 = \gamma_9 \, , \, b_2 = \gamma_{10} \tag{5.17}$$

then (5.14) holds.

We assume henceforth that b_1 and b_2 are fixed by (5.17) and define $\rho = \rho(t) \geq 0$ by

$$\rho^2 = \| \ddot{X} \|^2 + \| \dot{X} \|^2 + \| X \|^2 .$$

It is clear then from (5.14), (5.13), (5.5), (5.3) and (1.2) that

$$\gamma_4 \int_0^\omega \rho^2 \, dt \leq \gamma_{11} \int_0^\omega \rho dt + \varepsilon \gamma_{12} \int_0^\omega \rho^2 \, dt$$

for some γ_{11} and γ_{12}; so that if, for example,

$$\varepsilon \leq \tfrac{1}{2} \gamma_4 \gamma_{12}^{-1} \tag{5.18}$$

as we assume henceforth and $\gamma_{13} \equiv 2 \gamma_4^{-1} \gamma_{11}$ then

$$\int_0^\omega \rho^2 \, dt \leq \gamma_{13} \int_0^\omega \rho dt$$

$$\leq \gamma_{13} \, \omega^{\frac{1}{2}} \left(\int_0^\omega \rho^2 \, dt \right)^{1/}$$

by Schwarz's inequality. Hence

$$\left(\int_0^\omega \rho^2 \, dt \right)^{1/2} \leq \gamma_{13} \, \omega^{\frac{1}{2}}$$

that is

$$\int_0^\omega \rho^2 \, dt \leq \gamma_{14} \equiv \gamma_{13}^2 \, \omega \, . \tag{5.19}$$

The result (3.2) is a consequence of (5.19) as will now be shown. We begin by noting that (5.19) implies that

$$\int_0^\omega x_i^2 \, dt \leq \gamma_{14} \, , \int_0^\omega \dot{x}_i^2 \, dt \leq \gamma_{14} \, , \int_0^\omega \ddot{x}_i^2 \, dt \leq \gamma_{14} \qquad (i = 1 \, , 2 \, , \cdots , n) . \tag{5.20}$$

The inequality: $\int_0^\omega x_i^2 \, dt \leq \gamma_{14}$ here imples that $|x_i(\tau)| \leq \gamma_{15} \equiv \gamma_{14}^{\frac{1}{2}} \omega^{-\frac{1}{2}}$ for *some* $\tau \in [0, \omega]$, Thus, since $x_i(t) = x_i(\tau) + \int_\tau^t \dot{x}_i(s) \, ds$, we have at once that

$$\sup_{0 \leq t \leq \omega} |x_i(t)| \leq \gamma_{15} + \int_\tau^{\tau+\omega} |\dot{x}_i(s)| \, ds$$

$$\leq \gamma_{15} + \omega^{\frac{1}{2}} \left(\int_\tau^{\tau+\omega} \dot{x}_i^2(s) \, ds \right)^{1/2},$$

by Schwarz's inequality, which, in view of the second inequality in (5.20), leads in turn to the estimate

$$\sup_{0 \leq t \leq \omega} |\dot{x}_i(t)| \leq \gamma_{15} + \omega^{\frac{1}{2}} \gamma_{14}^{\frac{1}{2}} .$$

This is true for each $i = 1, 2, \cdots, n$ and hence

$$\|X\| \leq \gamma_{16} \, (0 \leq t \leq \omega) \tag{5.21}$$

for each ω-periodic solution $X(t)$ of (3.1) corresponding to $0 \leq \mu \leq 1$. Analagously the second and third inequalities in (5.20) also lead to the estimate

$$\sup_{0 \leq t \leq \omega} |\dot{x}_i(t)| \leq \gamma_{15} + \omega^{\frac{1}{2}} \gamma_{14}^{\frac{1}{2}} \qquad (i = 1, 2, \cdots, n)$$

which in turn implies that

$$\|\dot{X}\| \leq \gamma_{17} \qquad (0 \leq t \leq \omega) \tag{5.22}$$

for each ω-periodic solution $X(t)$ of (3.1) corresponding to $0 \leq \mu \leq 1$. It should be pointed out, however, that the middle inequality in (5.20), whose only role, as far as the verification of (5.22) is concerned, is to secure the existence of a $\tau \in [0, \omega]$ such that $|\dot{x}_i(\tau)| \leq \gamma_{15}$ is not actually crucial to the proof of (5.22) once the last inequality in (5.20) is available. This is because the existence of a $\tau \in [0, \omega]$ such that $|\dot{x}_i(\tau)| \leq \gamma$ for some γ is already a consequence of the ω-periodicity condition: $x_i(0) = x_i(\omega)$ which in fact implies that $\dot{x}_i(\tau_0) = 0$ for some $\tau_0 \in [0, \omega]$, so that because of the identity:

$$\dot{x}_i(t) = \dot{x}_i(\tau_0) + \int_{\tau_0}^t \ddot{x}_i(s) \, ds$$

we have that

$$\text{(5.23)} \qquad \sup_{0\le t\le\omega} |\dot{x}_1(t)| \le \omega^{\frac{1}{2}} \left(\int_{\tau_0}^{\tau_0+\omega} \ddot{x}_i^2(s)\, ds \right)^{1/2} \le \omega^{\frac{1}{2}} \gamma_{14}^{\frac{1}{2}}, \qquad (i = 1, 2, \cdots, n),$$

thus leading to: $\| \dot{X} \| \le \gamma \ (0 \le t \le \omega)$ as before.

To establish the last of the inequalities (3.2) it will suffice now to verify that

$$\text{(5.24)} \qquad \int_0^\omega \| \dddot{X} \|^2 \, dt \le \gamma_{18}$$

for any ω-periodic solution of (3.1) with $0 \le \mu \le 1$. For if indeed (5.24) holds, so that

$$\text{(5.25)} \qquad \int_0^\omega \dddot{x}_i^2 \, dt \le \gamma_{18} \qquad (i = 1, 2, \cdots, n),$$

then, since $\ddot{x}_i(\tau_1) = 0$ for some $\tau_1 \in [0, \omega]$ so that

$$\ddot{x}_i(t) = \ddot{x}_i(\tau_1) + \int_{\tau_1}^{t} \dddot{x}(s)\, ds = \int_{\tau_1}^{t} \dddot{x}(s)\, ds,$$

we shall have that

$$\text{(5.26)} \qquad \sup_{0\le t\le\omega} |\ddot{x}_i(t)| \le \omega^{\frac{1}{2}} \left(\int_{\tau_1}^{\tau_1+\omega} \dddot{x}^2(s)\, ds \right)^{1/2} \le \omega^{\frac{1}{2}} \gamma_{18}^{\frac{1}{2}} \qquad (i = 1, 2, \cdots, n),$$

by (5.25), which leads to the remaining estimate:

$$\text{(5.27)} \qquad \| \ddot{X} \| \le \gamma_{19} \qquad (0 \le t \le \omega)$$

in (3.2). As for the actual verification of (5.24) it is convenient to take a scalar product of either side of (3.1) with $\dddot{X}$ and integrate with respect to t from $t = 0$ to $t = \omega$. Since $X, \dot{X}$ are already subject to (5.21) and (5.22) and $\langle A\ddot{X}, \dddot{X} \rangle$ is a perfect t-differential, this integration shows readily in view of (1.2), that

$$\int_0^\omega \| \dddot{X} \|^2 \, dt \le \gamma_{20} \int_0^\omega \| \dddot{X} \| \, dt + \varepsilon \int_0^\omega \| \dddot{X} \| \cdot \| \ddot{X} \| \, dt$$

$$\le \left\{ \gamma_{10}\, \omega^{\frac{1}{2}} + \varepsilon \left(\int_0^\omega \| \ddot{X} \|^2 \, dt \right)^{1/2} \right\} \left(\int_0^\omega \| \dddot{X} \|^2 \, dt \right)^{1/2}.$$

Thus, since $\int_0^\omega \| \ddot{X} \|^2 \, dt \leq \gamma$, we must have that

$$\int_0^\omega \| \dddot{X} \|^2 \, dt \leq \gamma_{21} \left(\int_0^\omega \| \dddot{X} \|^2 \, dt \right)^{1/2} . \tag{5.28}$$

Hence

$$\int_0^\omega \| \dddot{X} \|^2 \, dt \leq \gamma_{21}^2$$

which is (5.24). Thus (5.27) holds if $\varepsilon \leq \frac{1}{2} \gamma_4 \gamma_{12}^{-1}$. This completely verifies the theorem with $\varepsilon_0 = \frac{1}{2} \gamma_4 \gamma_{12}^{-1}$.

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