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Stability of the abstract differential inequalities of the Cauchy type

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Equazioni differenziali ordinarie. — *Stability of the abstract differential inequalities of the Cauchy type.* Nota di OLUSOLA AKINYELE, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si trovano condizioni sufficienti per la stabilità e la stabilità asintotica delle soluzioni di una classe di disequazioni differenziali.

1. INTRODUCTION

Let $u(t)$ be a function from $R^+ = [0, \infty)$ into a Banach space Y with norm $\|\cdot\|$ and let it satisfy the differential inequality of the form.

$$(1) \qquad \left\| \frac{du}{dt} - A(t)\,u - f(t, u) \right\| \leq g_1(t, \|u\|), \; u(t_0) = u_0,$$

for $\|u\| < \rho$, where $f \in C(R^+ \times Y, Y)$, $f(t, 0) = 0$, $g, \in C(R^+ \times R^+, R^+)$ and $A(t)$ is a closed linear operator with dense domain and generally unbounded.

In this paper, employing the technique of differential inequalities and the comparison principle, under some conditions on $[I - hA(t)]^{-1}$ for sufficiently small $h > 0$, we give a set of sufficient conditions for the various types of stability properties of (1). Because of the obvious advantages derived from the use of several Lyapunov functions in obtaining stability results, we also give a set of sufficient conditions for the stability properties of the differential inequality (1) in terms of several Lyapunov functions. In addition we give sufficient conditions under which every solution $u(t)$ of (1) tends to zero as $t \to \infty$. Our results in this paper contain some known results as special cases (cf: 1, 3].

2. MAIN RESULTS

DEFINITION 2.1. Let $u(t)$ be a function defined and continuous for $t \geq t_0 \geq 0$. Suppose $u(t)$ is a strongly differentiable function such that $u(t) \in D(A(t))$ for $t \geq t_0$; and satisfies (1) for $t \in [t_0, \infty) \sim E$ where E is an at most countable subset of $[t_0, \infty)$. Then $u(t)$ is said to be a solution of the differential inequality (1). We shall assume throughout this paper, that the solution $u(t, t_0, u_0)$ of (1) exists for all $t \geq t_0$.

(*) Nella seduta del 10 febbraio 1979.

Remark. If $g_1(t, \| u \|) \equiv 0$, then E is an empty set and so $u(t)$ is a solution of the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \mathrm{A}(t)\, u + f(t, u) \,, \; u(t_0) = u_0 \,. \tag{2}$$

If $g_1(t, \| u \|) = \varepsilon$, then $u(t)$ is said to be an ε — approximate solution of the equation (2):

We list a few definitions concerning the stability properties of the differential inequality (1) with respect to the origin.

DEFINITION 2.2. The trivial solution of (1) is said to be

I_1: equistable if for each $\varepsilon > 0$ and $t_0 \in [0, \infty) \sim \mathrm{E}$ there exists a positive function $\delta = \delta(t_0, \varepsilon)$ that is continuous in t_0 for each ε such that $\| u_0 \| < \delta$ implies $\| u(t, t_0, u_0) \| < \varepsilon$ for $t \geq t_0$.

I_2: Unformly stable if I_1 holds with δ independent of t_0.

I_3: quasi–equi asymptotically stable if for each $\varepsilon > 0$ and $t_0 \in [0, \infty) \sim \mathrm{E}$ there exist positive numbers $\delta_0 = \delta_0(t_0)$ and $\mathrm{T} = \mathrm{T}(t_0, \varepsilon)$ such that $\| u_0 \| \leq \delta_0$ implies $\| u(t, t_0, u_0) \| < \varepsilon$ for $t \geq t_0 + \mathrm{T}$;

I_4: quasi-uniformly asymptotically stable if I_3 holds with the numbers δ_0 and T independent of t_0;

I_5: equi–asymptotically stable if I_1 and I_3 hold simultaneously;

I_6: Uniformly asymptotically stable if I_2 and I_4 hold simultaneously.

Consider the scalar differential equation.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g(t, x) \,, \qquad x(t_0) = x_0 \geq 0 \,, \tag{3}$$

where $g \in \mathrm{C}(\mathrm{R}^+ \times \mathrm{R}^+, \mathrm{R}^+)$, and $g(t, 0) = 0$, so that $x = 0$ is a solution of (3) through $(t_0, 0)$. This assumption implies that the solution $x(t, t_0, x_0)$ of (3) are non-negative for $t \geq t_0$. Corresponding to the stability definitions (I_1) to (I_6) we denote by (I_1^*) to $\mathrm{I}_6^*)$ the concepts of stability of the solution $x = 0$ of (3). We give one such definition since the remaining can be formulated similarly.

DEFINITION 2.3. The trivial solution of (3) is said to be:

I_1^*: equistable if for each $\varepsilon > 0$, $t_0 \in \mathrm{R}^+$ there exists a positive function $\delta = \delta(t_0, \varepsilon)$ that is continuous in t_0 for each ε such that $x(t, t_0, x_0) < \varepsilon$ $t \geq t_0$, provided $x_0 \leq \delta$.

DEFINITION 2.4. A function $\varphi(r)$ is said to belong to the class K if $\varphi \in \mathrm{C}([0, \rho), \mathrm{R}^+)$, $\varphi(0) = 0$ and $\varphi(r)$ is strictly monotone increasing in r.

We now give a set of sufficient conditions for the stability property of the differential inequality (1).

THEOREM 2.5. *Assume that*

(i) $V \in C(R^+ \times S_\rho, R^+)$, $V(t, 0) = 0$ *and*

$$|V(t, u_1) - V(t, u_2)| \leq L |u_1 - u_2|, (t, u_1), (t, u_2) \in R^+ \times S_\rho$$

where L *is a constant and* $S_\rho = \{u \in Y : \| u \| < \rho\}$;

(ii) $b(\| u \|) \leq V(t, u)$, $b \in K$, $(t, u) \in R^+ \times S_\rho$;

(iii) $g_2 \in C(R^+ \times R^+)$, $g_2(t, 0) = 0$ *and*

$$D^+ V(t, u)_{(2)} \leq g_2(t, V(t, u)), (t, u) \in R^+ \times S_\rho$$

where $D^+ V(t, u)_{(2)}$ *denotes the Dini derivative with respect to* (2);

(iv) $g_1(t, 0) = 0$ *and* $g_1(t, x)$ *is non decreasing in* x *for* $t \in R^+$,

(v) *For each* $t \in R^+$, *and all* $h > 0$ (h *small*) *the operator* $R[h, A(t)] = [I - hA(t)]^{-1}$ *exists as a bounded operator defined on* Y *and for each* $u \in Y$

$$\lim_{h \to 0} R[h, A(t)] u = u .$$

Then the stability properties I_1^* *to* I_6^* *of the trivial solution of* (3) *with*

$$g(t, x) = L g_1(t, b^{-1}(x)) + g_2(t, x) \tag{4}$$

imply the stability properties I_1 *to* I_6 *of the differential inequality* (1) *with respect to the trivial solution.*

Proof. Let $u(t)$ be any solution of (1) such that $V(t_0, u_0) \leq x_0$ and define

$$m(t) = V(t, u(t)),$$

then $m(t_0) \leq x_0$ and for small $h > 0$,

$$\begin{aligned} m(t+h) - m(t) &= V(t+h, u(t+h)) - V(t, u(t)) \\ &= V(t+h, u(t+h)) - V(t+h, R[h, A(t)] u + hf(t, u)) \\ &\quad + V(t+h, R[h, A(t)] u + hf(t, u)) - V(t, u) \\ &\leq L \| u(t+h) - \{R[h; A(t)] u + hf(t, u)\} \| \\ &\quad + (1/h) \{V(t+h, R[h; A(t)] u + hf(t, u) - V(t, u)\} . \end{aligned} \tag{5}$$

Since for every $u \in D(A(t)$, $R[h; A(t)][I - hA(t)] u = u$, it follows that

$$R[h; A(t)] u = u + hA(t) u + h \{R[h; A(t)] A(t) u - A(t) u\} . \tag{6}$$

Using (5) and (6),

$$\frac{m(t+h)-m(t)}{h} \leq L \left\| \frac{u(t+h)-u(t)}{h} - \{A(t)\,u + f(t,u)\} \right\|$$
$$+ L \| - R[h\,;A(t)]\,A(t)\,u + A(t)\,u \|$$
$$+ (1/h)\{V(t+h, R[h, A(t)]\,u + hf(t,u)) - V(y,u)\}.$$

Using the differential inequality (1), the assumptions (iii) and (v) and the monotonic character of $g_1(t, x)$ in x, we obtain

$$\begin{aligned} D^+ m(t) &\leq L g_1(t, \|u\|) + D^+ V(t, u)_{(2)} \\ &\leq L g_1(t, b^{-1}(V(t,u))) + g_2(t, V(t,u)) \\ &= L g_1(t, b^{-1}(m(t))) + g_2(t, m(t)) \\ &= g(t, m(t)) \end{aligned}$$

An application of Theorem 1.4.1 of [2] yields

$$V(t, u(t) \leq r(t, t_0, x_0) \qquad \text{for} \quad t \geq t_0 \tag{7}$$

where $r(t, t_0, x_0)$ is the maximal solution of (3) existing for $t \geq t_0$.

Now suppose I_1^* holds. Let $\varepsilon > 0$, and $t_0 \in R^+$ be given, then for $b(\varepsilon) > 0$ there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that $x_0 \leq \delta$ implies

$$x(t, t_0, x_0) < b(\varepsilon) \qquad t \geq t_0\,; \tag{8}$$

where $x(t, t_0, x_0)$ is any solution of (3). The local Lipschitzian property of $V(t, u)$ in u for constant $L > 0$ implies that

$$V(t_0, u_0) \leq L\|u_0\|, \qquad \text{since} \quad V(t, 0) = 0\,.$$

So choose $x_0 = L\|u_0\|$, then $V(t_0, u_0) \leq x_0$ and if we set $\delta_1(t_0, \varepsilon) = \frac{\delta(t_0, \varepsilon)}{L}$, then,

$$\|u(t, t_0, u_0)\| < \varepsilon \qquad t \geq t_0\,,$$

provided $\|u_0\| \leq \delta_1(t_0, \varepsilon)$ where $u(t, t_0, u_0)$ is any solution of (1). Suppose not; then for some $t_1 \geq t_0$, we have

$$\|u(t_1, t_0, u_0)\| = \varepsilon \quad \text{and} \quad \|u(t, t_0, u_0)\| \leq \varepsilon \quad \text{for} \quad t \in [t_0, t_1]\,.$$

Hence by (7)

$$b(\varepsilon) \leq V(t_1, u(t_1)) \leq r(t_1, t_0, x_0) < b(\varepsilon)\,,$$

which is a contradiction. So that I_1 holds for the differential inequality (1).

Suppose I_3^* holds, then let $\varepsilon > 0$, $t_0 \in R^+$ be given; then $\exists$ positive numbers $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, b(\varepsilon)) = T(t_0, \varepsilon) > 0$ such that $x_0 \leq \delta_0$ implies

$$x(t, t_0, x_0) < b(\varepsilon) \quad \text{for } t \geq t_0. \tag{9}$$

Since $V(t_0, u)$ is continuous and $V(t_0, 0) = 0$, there exists a $\delta_1 = \delta_1(t_0, \delta_0) < \delta_0$ such that $\| u_0 \| < \delta_1$ implies $V(t_0, u_0) < \delta_0$. Now let $\| u_0 \| < \delta_1$ and $t \geq t_0 + T$, then

$$b(\| u(t, t_0, u_0) \|) \leq V(t, u(t, t_0, u_0)) \leq r(t, t_0, x_0) < b(\varepsilon)$$

and so $\| u(t, t_0, u_0) \| < \varepsilon$ for $t \geq t_0 + T$, which is I_3. By choosing δ and T appropriately and combining the results for I_1 and I_3 the rest of the proof can be constructed on the basis of the inequality (7). By choosing δ and T independent of t_0 I_2^* I_4^* and I_6^* imply I_2 I_4 and I_6.

In the next theorem we give a set of conditions for the stability properties of the differential inequality (1) in terms of several Lyapunov functions, since using several Lyapunov functions is more advantageous and leads to a more flexible mechanism. Moreover for several Lyapunov functions each function could satisfy less rigid requirements.

THEOREM 2.6. *Assume that*

(i) $V \in C(R^+ \times S_\rho, R^n)$, $V(t, 0) = 0$ *and*

$$\| V(t, u_1) - V(t, u_2) \| \leq L \| u_1 - u_2 \|, (t, u_1), (t, u_2) \in R^+ \times S_\rho$$

where L *is a constant*;

(ii) $b(\|u\|) \leq \sum_{i=1}^{n} V_i(t, u)$, $b \in K$, $(t, u) \in R^+ \times S_\rho$, *and* $\sum_{i=1}^{n} V_i(t, u) \to 0$ *as* $\|u\| \to 0$ *for each* $t \in R^+$,

(iii) $g_2 \in C(R^+ \times R^n, R^n)$, $g_2(t, 0) = 0$, $g(t, x)$ *is :–quasi-monotone nondecreasing in* x *for each* $t \in R^+$ *and*

$$D^+ V(t, u)_{(2)} \leq g_2(t, V(t, u) \qquad (t, u) \in R^+ \times S_\rho;$$

(iv) $g_1(t, 0) = 0$ *and* $g_1(t, x)$ *is quasi-monotone nondecreasing in* x *for each* $t \in R^+$;

(v) *For each* $t \in R^+$, *all* $h > 0$ *(h small) the operator* $R[h, A(t)]$ *exists as a bounded operator defined on* Y *and for each* $u \in Y$.

$$\lim_{h \to 0} R[h, A(t)] u = u$$

Then the stability properties I_1^*, I_3^* *and* I_5^* *of the trivial solution of the auxilliary differential system*

$$\frac{dw}{dt} = g(t, w) \tag{10}$$

with

$$g(t, w) = \mathrm{L} g_1(t, b^{-1}(w)) + g_2(t, w)$$

and maximal solution $r(t, t_0, w_0)$ *existing for* $t \geq t_0$, *imply the stability properties* I_1, I_3 *and* I_5 *of the trivial solution of the inequality* (1).

Proof. Let $u(t) = u(t, t_0, u_0)$ be a solution of (1) such that $\mathrm{V}(t_0, u_0) \leq w_0$ and define the vector function

$$m(t) = \mathrm{V}(t, u(t, t_0, u_0)) \qquad t \geq t_0 .$$

Proceeding as in the last theorem together with the assumptions we obtain

$$\mathrm{D}^+ m(t) \leq g(t, m(t)) \qquad \text{for} \quad t \geq t_0 .$$

An application of corollary 1.7.1 of [2] yields

$$\mathrm{V}(t, u(t, t_0, u_0)) \leq r(t, t_0, w_0) \qquad \text{for} \quad t \geq t_0 . \tag{11}$$

Suppose I_1^* holds. Let $\varepsilon > 0$ and $t_0 \in \mathrm{R}^+$, then for $b(\varepsilon) > 0$. There exists $\delta = \delta(t_0, \varepsilon) > 0$ such that $\sum_{i=1}^{n} w_{i0} \leq \delta$ implies

$$\sum_{i=1}^{n} w_i(t, t_0, w_0) < b(\varepsilon) \qquad t \geq t_0 \tag{12}$$

where $w(t, t_0, w_0)$ is any solution of (10) for $t \geq t_0$.

Choosing $w_{i0} = \mathrm{V}_i(t_0, u_0)$ $i = 1, 2, \cdots, n$, using the hypothesis (ii) and proceeding as in Theorem 2.5, the we obtain I_1 for the differential inequality (1).

Suppose I_3^* holds then let $\varepsilon > 0$, $t_0 \in \mathrm{R}^+$ be given, then there exist positive numbers $\delta_0 = \delta_0(t_0)$ and $\mathrm{T} = \mathrm{T}(t_0, \varepsilon) > 0$ such that $\sum_{i=1}^{n} w_{i0} \leq \delta_0$ implies

$$\sum_{i=1}^{n} w_i(t, t_0, w_0) < b(\varepsilon), \qquad t \geq t_0 . \tag{13}$$

By hypothesis (ii) there exists a $\delta_1 = \delta_1(t_0, \delta_0) < \delta_0$ such that

$$\| u_0 \| \leq \delta_1 \quad \text{implies} \quad \sum_{i=1}^{n} \mathrm{V}_i(t_0, u_0) < \delta^0 .$$

Now let $\| u_0 \| \leq \delta_1$ and set $w_{i0} = \mathrm{V}_i(t_0, u_0)$ $i = 1, 2, \cdots, n$, then for $t \geq t_0 + \mathrm{T}$,

$$b(\| u(t, t_0, u_0) \| \leq \sum_{i=1}^{n} \mathrm{V}_i(t, u(t, t_0, u_0)) < \sum_{i=1}^{n} r_i(t, t_0, w_0) < b(\varepsilon) .$$

So $\| u(t, t_0, u_0) \| < \varepsilon$ for $t \geq t_0 + \mathrm{T}$ which is I_3.

THEOREM 2.7. *Assume that hypothesis* (i), (ii), (iii), (iv) *and* (v) *of Theorem* 2.6 *hold and* $\sum_{i=1}^{n} V_i(t, u) \to 0$ *as* $\|u\| \to 0$ *uniformly in* $t \in R^+$.

Then the stability properties I_2^*, I_4^* *and* I_6^* *of the trivial solution of the auxiliary differential system* (10) *with*

$$g(t, w) = Lg_1(t, b^{-1}(w)) + g_2(t, w)$$

and maximal solution $r(t, t_0, w_0)$ *existing for* $t \geq t_0$, *imply the stability properties* I_2, I_4 *and* I_6 *of the trivial solution of the differential inequality* (1).

Proof. By choosing δ, δ_1 and T of the last theorem independet of t_0 the results can be established. We leave the details.

THEOREM 2.8. *In addition to the hypotheis* (ii), (iii), (iv) *and* (v) *of Theorem* 2.5 *assume that* $V \in C(R^+ \times S_\rho, R^+)$ $V(t, 0) \equiv 0$ *and*

$$|V(t, u_1) - V(t, u_2)| \leq L(t) \|u_1 - u_2\|$$

where (t, u_1), $(t; u_1)$, $(t, u_2) \in R^+ \times S_\rho$ *and* $L(t) \geq 0$ *is a continuous function of* t *on* R^+.

Then the stability properties I_1^*, I_3^* *and* I_5^* *of the trivial solution of* (3) *with*

$$g(t, x) = L(t) g_1(t, b^{-1}(x)) + g_2(t, x)$$

imply the stability properties I_1, I_3 *and* I_5 *respectively of the differential inequality* (1), *with respect to the origin.*

Proof. Define

$$m(t) = V(t, u(t, t_0 u_0)) \quad t \geq t_0$$

and proceed as in Theorem 2.5 to obtain the inequality

$$V(t, u(t, t_0, u_0)) \leq r(t, t_0, x_0) \qquad \text{for} \quad t \geq t_0 .$$

With this inequality the remainder of the proof follows closely that of Theorem 5.7.2 of [1].

THEOREM 2.9. *If in addition to the hypothesis of Theorem* 2.8 *we assume that there exist a function* $\alpha : R^+ \to R^+$ *such that* $\alpha(r)$ *is increasing in* r *and*

$$V(t, u(t, t_0, u_0)) \leq \alpha(\|u\|)\ (t, u) \in R^+ \times Y .$$

Then the stability properties I_2^*, I_4^* *and* I_6^* *of the trivial solution of* (3) *with*

$$g(t, w) = L(t) g_1(t, b^{-1}(w)) + g_2(t, w)$$

imply the stability properties I_2, I_4 *and* I_6 *respectively of the differential inequality* (1) *with respect to the origin.*

Proof. Since

$$V(t, u(t, t_0, u_0)) \leq \alpha(\|u\|), (t, u) \in R^+ \times Y,$$

then δ and T which would emerge in the proof of Theorem 2.8 can be chosen appropriately to be independent of t_0, so that the proof follows closely on the proof of Theorem 2.8.

Remarks. Our results contain many special cases. If $g_1(t, u) = 0$, we obtain the stability results for the trivial solution of the abstract differential system (2) [1, Theorem 5.7.2].

If $R(t, u) \in C(R^+ \times Y, Y)$ and for $\|u\| < \rho$,

$$\|R(t, u)\| \leq g_1(t, \|u\|),$$

then we deduce from our results the stability properties of the trivial solution of the abstract system (2) with respect to permanent perturbations $R(t, u)$.

If $R(t, u) = \int_{t_0}^{t} k(t, s)\, g(s, u(s))\, ds$, $u(t_0) = u_0$, where $u \in Y$, $g \in C(R^+ \times Y, Y)$, $g(t, 0) \equiv 0$, $k(t, s) \in C(R^+ \times R^+, R^+)$ and $\int_{t_0}^{t} k(t, s) \|g(s, u(s))\| ds \leq g_1(t, \|u\|)$, the uniform asymptotic stability result of the trivial solution of the integro differential system

$$(**) \qquad \frac{du}{dt} = A(t)\, u + f(t, u) + \int_{t_0}^{t} k(t, s)\, g(s, u(s))\, ds, \qquad u(t) = u_0,$$

discussed in [3, Theorem 2] is a special case of our Theorem 2.5. We also note that other stability properties of the system (**) could be deduced from our results in Theorems 2.5, 2.6, 2.7, 2.8 and 2.9. Such stability results are new and analogous to the corresponding results for integrodifferential systems (**) in Euclidean spaces.

The following theorem gives a set of conditions under which every solution of the differential inequality (1) tends to zero as $t \to \infty$.

THEOREM 2.10. *Assume that hypotheses* (i), (ii), (iii), (iv) *and* (v) *of Theorem* 2.5 *hold. Assume that the solutions* $x(t, t_0, x_0)$ *of* (3) *with* $g(t, x)$ *given by* (4) *for* $0 \leq x_0 \leq \beta$ *have the property that*

$$\lim_{t \to \infty} x(t, t_0, x_0) = 0.$$

Then every solution of the differential inequality (1) *starting in the set*

$$A = \{u \in Y : V(t, u) \leq \beta, t \geq 0\},$$

tends to zero as $t \to \infty$.

Proof. Let $u(t, t_0, u_0)$ be any solution of the inequality (1) such that $u_0 \in \mathrm{A}$. Define

$$m(t) = \mathrm{V}(t, u(t)).$$

By repeating the same types of argiments as in Theorem 2.5,

$$\mathrm{D}^+ \mathrm{V}(t, u(t)) \leq g(t, \mathrm{V}(t, u(t))), \qquad t \geq t_0,$$

for $(t, u) \in \mathrm{R}^+ \times \mathrm{Y}$ and so again

$$\mathrm{V}(t, u(t, t_0, u_0)) \leq r(t, t_0, \beta), \qquad \text{for} \quad t \geq t_0,$$

where $r(t, t_0, \beta)$ is the maximal solution of (3) with $x_0 = \beta$. Hence

$$b(\| u(t, t_0, u_0) \|) \leq \mathrm{V}(t, u(t, t_0, u_0)) \leq r(t, t_0, \beta).$$

Since $\lim\limits_{t \to \infty} r(t, t_0, \beta) = 0$, $b(\| u(t, t_0, u_0) \|) \to 0$ as $t \to \infty$. Hence $\| u(t, t_0, u_0) \| \to 0$ as $t \to \infty$.

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