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Some results on a class of degenerate evolution problems

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Equazioni funzionali. — *Some results on a class of degenerate evolution problems.* Nota di ANGELO FAVINI (*), presentata (**) dal Corrisp. G. CIMMINO.

RIASSUNTO. — Si considera una equazione di evoluzione della forma

$$\frac{d}{dt} Bx(t) = -Ax(t) + f(t), \; t > 0, \qquad \| Bx(t) - z_0 \|_X \xrightarrow[t \to 0+]{} 0,$$

dove la funzione data f e la funzione incognita $x(\cdot)$ sono a valori negli spazi di Banach complessi X e Y rispettivamente, A e B sono operatori lineari chiusi da Y a X e B può non avere inverso limitato. Sono date condizioni sul risolvente B–modificato di A, che forniscono l'esistenza e l'unicità delle soluzioni. I risultati sono applicati ad alcune classi di equazioni degeneri alle derivate parziali.

1. INTRODUCTION

This note is devoted to the abstract initial value problem

$$\frac{d}{dt} Bx(t) = -Ax(t) + f(t), \; t \in (0, +\infty) = R^+, \tag{1.1}$$

$$\lim_{t \to 0+} \| Bx(t) - z_0 \|_X = 0, \tag{1.2}$$

where A, B are linear closed operators from the complex Banach space Y into the Banach space X, with their domains $\mathcal{D}(A)$, $\mathcal{D}(B)$ respectively, $f(\cdot)$ is a strongly continuous function from $[0, +\infty) = \overline{R}^+$ into X and $z_0 \in X$ is given.

We remark that the "solution" $x(\cdot)$ of (1.1) (1.2) may have either no strong derivative or no limit, in whatever sense, as $t \to 0+$. Hence, we shall also analyse the Cauchy problem

$$B \frac{dx(t)}{dt} \left(= B\dot{x}(t)\right) = -Ax(t) + f(t), \; t \in (0, +\infty), \tag{1.3}$$

$$\lim_{t \to 0+} \| x(t) - x_0 \|_Y = 0. \tag{1.4}$$

A detailed study of (1.3) in the finite-dimensional setting is to be found, for example, in [G 1] and [C 1].

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In very recent times other papers and books have appeared in the literature which consider (1.3) (1.4) in the abstract case and under various hypotheses; see [B 1, C 2, C 3, D 1, S 1, Z 1].

Here we follow an approach based on Laplace transform method and thus, the location of the set of all complex numbers λ for which $\lambda B + A$ has a bounded inverse $(\lambda B + A)^{-1} = R_B(\lambda ; A)$ plays a primary role. We shall see that the request for sufficiently smooth solutions involves somewhat restrictive assumptions on the data.

A detailed version with proofs of these results is to be found in [F 1].

2. DEFINITIONS. *Let m be a non negative integer; we shall denote by* $C^{(m)}(R^+ ; X)$ *(respectively,* $C^{(m)}(\bar{R}^+ ; X)$*), the set of all* X*-valued functions defined and m-times strongly continuously differentiable on* R^+ (resp., $\bar{R}^+$).

If f is an element of $C^{(0)}(\bar{R}^+ ; X)$, *the* Y*-valued function* $x(\cdot)$ *will be called a* strong *solution of* (1.1) (1.2) *if* $x(\cdot) \in C^{(0)}(R^+ ; Y)$, $x(t) \in \mathscr{D}(A) \cap \mathscr{D}(B)$ *for all* $t \in R^+$, $Ax(\cdot) \in C^{(0)}(R^+ ; X)$, $Bx(\cdot) \in C^{(1)}(R^+ ; X)$ *and* (1.1) (1.2) *are verified.*

$x(\cdot)$ *will be said to be a* classical *solution of* (1.3) (1.4) *if* $x(\cdot) \in C^{(0)}(\bar{R}^+ ; Y) \cap C^{(1)}(R^+ ; Y)$, $x(t) \in \mathscr{D}(A)$, $\dot{x}(t) \in \mathscr{D}(B)$ *for every* $t \in R^+$, $Ax(\cdot) \in C^{(0)}(R^+ ; X)$, $B\dot{x}(\cdot) \in C^{(0)}(R^+ ; X)$ *and* (1.3) (1.4) *hold.*

3. *Uniqueness and existence of the solutions.* Making use of an argument as in [K 2, p. 63], we have the following result about existence of a solution:

THEOREM 3.1. *Suppose that* $R_B(\lambda ; A)$ *exists for all sufficiently large real* λ *and there are* $C \in R^+$, *a non negative integer k such that for the norm of* $AR_B(\lambda ; A)$ *as a bounded operator from* X *into itself, we have* $\| AR_B(\lambda ; A) \|_{X \to X} \leq C\lambda^k$ *for these* λ*'s. Then the strong solution of* (1.1) (1.2) *is unique.*

Remark 3.1. By a standard change-of-variable argument we may replace A by $A + \lambda_2 B$ where $\lambda_2 > \lambda_1$, in (1.1) or (1.3). Therefore we may assume without loss of generality that A has a bounded inverse and $\mathscr{D}(A) \subseteq \mathscr{D}(B)$. This assumption will be kept henceforth.

As regards existence, we have

THEOREM 3.2. *Suppose that* $\lambda B + A$ *has a bounded inverse for all complex numbers* λ *such that* $\mathrm{Re}\,\lambda \geq \alpha > 0$ *and*

$$\| AR_B(\lambda ; A) \|_{X \to X} \leq C(|\lambda|^k + 1), \ \mathrm{Re}\,\lambda \geq \alpha, \tag{3.1}$$

where K *is a non negative integer and* C *is a positive constant.*

If $z_0 = B(A^{-1}B)^{k+1} z_1$; $f \in C^{(k+2)}(\bar{R}^+ ; X)$, $f^{(j)}(0) = \frac{d^j}{dt^j} f(0) = B(A^{-1}B)^{k-j} w_j$, $j = 0, 1, \cdots, k$, with $z_1, w_j \in D(A)$, *then* (1.1) (1.2) *has the unique strong solution*

$$x(t) = \sum_{j=0}^{k+1} (-1)^j \frac{t^j}{j!} (BA^{-1})^{k-j+1} z_1 + (-1)^{k+2} Z_{k+2}(t) A z_1 +$$

$$+ \sum_{j=0}^{k} \left\{ \sum_{s=0}^{k-j+1} (-1)^{s+1} \frac{t^{j+s}}{(j+s)!} (BA^{-1})^{k-j-s+1} w_j + (-1)^{k-j+3} Z_{k+2}(t) A w_j \right\} +$$

$$+ Z_{k+2}(t) f^{(k+1)}(0) + \int_0^t Z_{k+2}(t-s) f^{(k+2)}(s)\, ds\,,\ t \in R^+\,,$$

where

$$Z_p(t) = (2\pi i)^{-1} \int_{\alpha_1 - i\infty}^{\alpha_1 + i\infty} \lambda^{-p} e^{\lambda t} R_B(\lambda; A)\, d\lambda\,,\ p \geq k+2\,,\ t \in R^+\,,$$

and $\alpha_1 > \alpha$.

Remark 3.2. We want to give a simple example to emphasize that z_0 and f must satisfy some compatibility and regularity conditions so that the initial value problem can have a solution or be well-posed.

Let $z_0 = (x_1, y_1)$, $f = (f_1, f_2)$ be elements of $\mathbf{C}^2 = \mathbf{C} \times \mathbf{C}$ and of $C^{(0)}(\bar{R}^+; \mathbf{C}^2)$, respectively. Define A, B as the linear operators associated to the matrices (a_{ij}), (b_{ij}), $a_{11} = a_{22} = 1$, $a_{12} = a_{21} = 0$, $b_{11} = b_{21} = b_{22} = 0$, $b_{12} = 1$, respectively.

Then (1.1) (1.2) has a strong solution if we assume that $f_2 \in C^{(1)}(\bar{R}^+; \mathbf{C})$ and $x_1 = f_2(0)$, $y_1 = 0$, since $x(t) = f_1(t) - f_2(t)$, $y(t) = f_2(t)$.

THEOREM 3.3. *Suppose that* (3.1) *holds. If*

$$x_0 = (A^{+1} B)^{k+2} x_1\,,\quad x_1 \in \mathcal{D}(A)\,,\quad f \in C^{(k+3)}(\bar{R}^+; X)\,,$$

$$f^{(j)}(0) = B (A^{-1} B)^{k+1-j} z_j\,,\quad z_j \in \mathcal{D}(A)\,,\quad j = 0, 1, \cdots, k+1,$$

then (1.3) (1.4) *has a classical solution.*

If (3.1) is satisfied with $k = 0$ on all of $\mathbf{C}^+$, the set of all complex numbers with positive real parts, it is possible to obtain a refinement of Theorem 3.2. In fact, we are in a nearly-analytic case.

It is not too difficult to see that $R_B(\lambda; A)$ has then an analytic extension on $S_\omega : |\arg \lambda| < \omega$, $\pi/2 < \omega < \pi$, and an estimate as (3.1) is retained on $S_{\omega-\varepsilon}$. Define

$$Z_j(t) = (2\pi i)^{-1} \int_\Gamma e^{\lambda t} \lambda^{-j} R_B(\lambda; A)\, d\lambda\,,\qquad j = 0, 1, \cdots,$$

where Γ is a suitable contour in $S_{\omega-\varepsilon}$, $0 < \varepsilon < \omega - \pi/2$, coinciding with the rays $|\arg \lambda| = \omega - \varepsilon$, for $|\lambda| >$ const., and avoinding the origin to the right. We have

THEOREM 3.4. *Suppose that the foregoing hypotheses are satisfied. If* z_0 *belongs to the closure of* $B(\mathcal{D}(A))$ *and* $f \in C^{(2)}(\overline{R}^+; X)$, *then* (1.1) (1.2) *has the unique strong solution*

$$x(t) = Z_0(t) z_0 + \sum_{j=1}^{2} Z_j(t) f^{(j-1)}(0) + \int_0^t Z_2(t-s) f^{(2)}(s)\, ds, \quad t \in R^+.$$

If $Ax_0 = Bx_1$, $x_1 \in \mathcal{D}(A)$, $f \in C^{(3)}(\overline{R}^+; X)$, $f(0) = B\omega$, $\omega \in \mathcal{D}(A)$, *then*

$$x(t) = x_0 + t(\omega - x_1) - Z_2(t)(\omega - x_1) + \sum_{j=2}^{3} Z_j(t) f^{(j-1)}(0) + \int_0^t Z_3(t-s) f^{(3)}(s)\, ds$$

is a classical solution of (1.3) (1.4).

Next theorem concerns the finite dimensional case. By use of the spectral representation of a linear operator as given in [K 1, p. 41], we obtain

THEOREM 3.5. *Suppose that* $X = Y = \mathbf{C}^n$ *and* $f \in C^{(m_1-1)}(\overline{R}^+; X)$, *where* m_1 *is the algebraic multiplicity of* 0 *as an eigenvalue of* BA^{-1}.

Let $\lambda_1 = 0, \lambda_2, \cdots, \lambda_s$, *be the distinct eigenvalues of* BA^{-1} *and define*

$$Q_k = -(2\pi i)^{-1} \int_{\gamma_k} (zA + B)^{-1}\, dz, \quad k = 1, \cdots, s,$$

where γ_k *is a positively-oriented small circle enclosing* $-\lambda_k$ *and excluding* $-\lambda_j$, $j \neq k$. *Then the general solution of* (1.1) *is given by*

$$x(t) = \sum_{j=0}^{m_1-1} (-1)^j A^{-1} (BQ_1)^j AQ_1 f^{(j)}(t) +$$
$$+ \sum_{k=2}^{s} \{A^{-1} e^{-t(BQ_k)^{-1}} AQ_k w + \int_0^t A^{-1} e^{-(t-s)(BQ_k)^{-1}} (BQ_k)^{-1} AQ_k f(s)\, ds\}, \quad w \in X.$$

In particular, if $m_1 = 1$, *then* (1.1) (1.2) *has a unique strong solution for all* z_0 *in the range of* B *and* $f \in C^{(0)}(\overline{R}^+; X)$.

4. *Applications*. Let $A(x, D)$, $B(x, D)$ be strongly elliptic operators in a bounded domain Ω of R^n with coefficients continuous in $\overline{\Omega}$ and let $\partial\Omega$ be of class C^{2m}:

$$A(x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha, \quad B(x, D) = \sum_{|\beta| \leq 2p} b_\beta(x) D^\beta.$$

Suppose $p < m$ and $A(x, D)$ uniformly strongly elliptic.

If A, B are the operators defined in $L^q(\Omega)$, $q \geq 1$, by $(Au)(x) = A(x, D) u(x)$, $(Bv)(x) = B(x, D) v(x)$, $u \in \mathcal{D}(A) = H^{2m,q}(\Omega; \{B_j\})$, $v \in \mathcal{D}(B) = H^{2p,q}(\Omega; \{C_j\})$, [F 2, p. 75], and $(A, \{B_j\}, \Omega)$ is a regular elliptic boundary problem, there is $\varepsilon_1 \in R^+$ such that for all $0 < \varepsilon < \varepsilon_1$, the estimate

$$\| B(x, D) u \|_{L^q(\Omega)} \leq \varepsilon \| A(x, D) u \|_{L^q(\Omega)} + C \varepsilon^{-p(m-p)^{-1}} \| u \|_{L^q(\Omega)}, \quad u \in C^{(2m)}(\overline{\Omega})$$

holds.

On the other hand, we have

LEMMA 4.1. *Suppose that* $\| R_B(\lambda; A) \|_{X \to Y} \leq C |\lambda|^l$, $Re\,\lambda \geq \alpha$, *where* l *is a non negative integer. If there are a positive integer* m *and* $C \in R^+$ *such that for all* $\varepsilon \in R^+$

$$\| Bx \|_X \leq \varepsilon \| Ax \|_X + C\varepsilon^{-m} \| x \|_Y, \; x \in D(A),$$

then $\| AR_B(\lambda; A) \|_{X \to X} \leq C |\lambda|^{l+m+1}$, $Re\,\lambda \geq \alpha$.

LEMMA 4.2. *Let* $X = Y$ *be and suppose that there are a positive function* $C(\lambda)$ *on* $Re\,\lambda \geq \alpha > 0$ *and a domain* W *of* $\mathbf{C}$ *containing the origin, so that*

$$\| (\mu + \lambda B + A) x \|_X \geq C(\lambda) \| x \|_X, \; x \in D(A), \; \mu \in W. \tag{4.1}$$

Further, for any $Re\,\lambda \geq \alpha$ *we assume that a* $\mu(\lambda) \in W$ *can be found such that* $\mu(\lambda) + \lambda B + A$ *has a bounded inverse.*

Then $R_B(\lambda; A)$ *exists for* $Re\,\lambda \geq \alpha$ *and* $\| R_B(\lambda; A) \|_{X \to Y} \leq C(|\lambda|)^{-1}$.

For example, if $X = Y$ is a Hilbert space with inner product $\langle , \rangle$, and $\langle Bx, x \rangle \geq 0$, $Re \langle Ax, x \rangle \geq m_0 \| x \|^2$, $m_0 \in R^+$, $x \in \mathscr{D}(A)$, then (4.1) is satisfied with $W = \{\mu \in \mathbf{C}; Re\,\mu > -m_0/2\}$ and $C(\lambda) = m_0/2$.

In view of these results, we can handle problems of the type

$$\frac{\partial}{\partial t} B(x, D) u(t, x) + A(x, D) u(t, x) = f(t, x), \; (t, x) \in R^+ \times \Omega,$$

$$B_j(x, D) u = 0, \; x \in \partial\Omega, \; 1 \leq j \leq m,$$

$$\lim_{t \to 0+} B(x, D) u(t, x) = v_0(x), \; x \in \Omega.$$

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