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On finite dimensional spaces of measurable functions

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Analisi funzionale. — *On finite dimensional spaces of measurable functions.* Nota di MAZEN SHAHIN, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Com'è noto, un operatore autoaggiunto (o più in generale normale) in uno spazio di Hilbert è unitariamente equivalente all'operatore di moltiplicazione per una certa funzione $F : X \to \mathbf{C}$ sul corrispondente spazio $L_2(X, dx)$.

Hanno interesse quelle perturbazioni di un dato operatore autoaggiunto per le quali muta non solo il modo di trasformarsi sotto l'azione dell'operatore, ma anche il suo dominio di definizione (in particolare ciò significa perturbazione delle condizioni iniziali). Perciò, grosso modo, ci si restringe a domini di definizione chiusi nel senso di Mackey.

1. INTRODUCTION

Let R denote the set of real numbers and C the set of complex numbers. Let M (R) be the space of equivalence classes of complex-valued measurable functions on R, under the equivalence relation: f equivalent to g means that $f(x) = g(x)$ a.e. on R. We often write "function" when we mean an element of M (R). Let H denote the Hilbert space $L_2(R)$. By $(\cdot\,,\cdot)$ we denote the scalar product in H. For $\mathscr{F} \in M(R)$ we denote by $\mathscr{F}(\tilde{\mathscr{X}})$ the mapping of $M(R) \to M(R)$ defined by

$$\mathscr{F}(\tilde{\mathscr{X}})\, f(x) = \mathscr{F}(x) f(x) \qquad \text{a.e. on } R,$$

and we let $\mathscr{F}(\mathscr{X})$ be the restriction of $\mathscr{F}(\tilde{\mathscr{X}})$ to H,

$$D(\mathscr{F}(\mathscr{X})) = \{f \in H : \mathscr{F}(\tilde{\mathscr{X}}) f \in H\}$$

We introduce the following notation:

If N is a subset of M (R), then

$$N^{\perp} = \left\{ f \in L_2(R) : \forall g \in N \quad fg \in L_1(R) \quad \text{and} \quad \int_R f\bar{g}\, dx = 0 \right\}.$$

If G is a subset of $L_2(R)$, then

$$G^{\mathsf{T}} = \left\{ f \in M(R) : \forall g \in G \quad fg \in L_1(R) \quad \text{and} \quad \int_R f\bar{g}\, dx = 0 \right\}.$$

We say that $U \subset M(R)$ is closed in the Mackey sense, or simply closed, if it coincides with its bi-annihilator (see [2] p. 80, [3]). Our purpose here is to study this property of finite dimensional spaces of measurable functions. Our main result is theorem 2.1.

(*) Nella seduta del 13 gennaio 1979.

2. Some properties of finite dimensional subspaces

Proposition 2.1. *Let* $u_1, \cdots, u_n \in \mathrm{M}(\mathrm{R})$ *and be linearly independent. Let* $\mathscr{E}_1 \subset \mathscr{E}_2 \subset \cdots \subset \mathrm{R}$ *such that* $\mathrm{R} \setminus \bigcup_\nu \mathscr{E}_\nu$ *has measure zero. Then beginning with some* ν *the functions* $u_1, \cdots, u_n$ *are linearly independent on* $\mathscr{E}_\nu$.

Proof. By contradiction. Suppose that there exist numbers

$$c_1^{(\nu)}, \cdots, c_n^{(\nu)}, \qquad \text{such that}$$

$$|c_1^{(\nu)}|^2 + \cdots + |c_n^{(\nu)}|^2 = 1 ,$$

and a.e. on $\mathscr{E}_\nu$

$$\sum_j c_j^{(\nu)} u_j(x) = 0$$

If $(c_1, \cdots, c_n)$ is a limit point of the sequence

$$\{(c_1^{(\nu)}, \cdots, c_n^{(\nu)})\}_\nu^\infty = 1 , \qquad \text{then}$$

$$\sum_j c_j u_j(x) = 0 \qquad \text{a.e.} \quad \text{on} \quad \mathrm{R}.$$

Proposition 2.2. *Let* U *be a finite dimensional subspace of the (linear) space* M (R). *Let* $\mathscr{E}_1, \mathscr{E}_2, \cdots$ *be a sequence of sets as in proposition* 2.1., *and* f *a function* $\mathrm{R} \to \mathrm{C}$, *such that for every* ν *there exists a function* u_ν *from* U *such that the restriction of* f *to* $\mathscr{E}_\nu$ *is equal to the restriction of* u_ν *to* $\mathscr{E}_\nu$. *Then* $f \in \mathrm{U}$.

Proof. Let $u_1, \cdots, u_n$ be a basis of U. The functions $f, u_1, \cdots, u_n$ are linearly dependent on $\mathscr{E}_\nu$ $(\nu = 1, 2, \cdots)$. From proposition 2.1. it follows that $f, u_1, \cdots, u_n$ are linearly dependent on R.

Proposition 2.3. *Let* U *be as in proposition* 2.2. *Then there exists a sequence* $\mathscr{E}'_1 \subset \mathscr{E}'_2 \subset \cdots$ *such that* $\mathscr{E}'_1 \subset \mathscr{E}'_2 \subset \cdots \subset \mathrm{R}$, $\mathrm{R} \setminus \bigcup_\nu \mathscr{E}'_\nu$ *is a set of measure zero and every function* f *in* U *is bounded on* $\mathscr{E}'_\nu$.

Proof. If $u_1, \cdots, u_n$ is a basis for U, then it is sufficient to take

$$\mathscr{E}'_\nu = \{x \in \mathrm{R} : \max_j |u_j(x)| < \nu\} .$$

Lemma 2.1. *Let* $\mathrm{U} \subset \mathrm{M}, \mathrm{U} \cap \mathrm{H} = (0)$, *dim* $\mathrm{U} < \infty$, $\mathscr{F} \in \mathrm{M}$, *Then* $\mathrm{D}(\mathscr{F}(\mathscr{X})) \cap \mathrm{U}^\perp$ *is dense in* H.

Proof. Let $\mathscr{E}'_\nu$ be as in proposition 2.3., and let

$$\mathscr{E}_\nu = \{x \in \mathrm{R} : |x| < \nu , |\mathscr{F}(x)| < \nu\} \cap \mathscr{E}'_\nu .$$

Let χ_ν be the characteristic function of $\mathscr{E}_\nu$ and

$$\mathrm{H}_\nu = \chi_\nu(\mathscr{X}) \mathrm{H} .$$

We have

$$\chi_\nu(\tilde{\mathscr{X}})\, U \subset H_\nu \subset H ,$$

hence

$$H_\nu \cap U^\perp = H_\nu \ominus \chi_\nu(\tilde{\mathscr{X}})\, U . \tag{2.1}$$

Let $f \in H$ and $f \perp D(\mathscr{F}(\mathscr{X})) \subset U^\perp$.
Then $f \perp H_\nu \cap U^\perp$, since $H_\nu \subset D(\mathscr{F}(\mathscr{X}))$.
Consequently

$$\chi_\nu(\mathscr{X}) f \perp (H_\nu \cap U^\perp)$$

and by virtue of (2.1), we get

$$\chi_\nu(\mathscr{X}) f \in \overline{\chi_\nu(\mathscr{X})\, U} = \chi_\nu(\mathscr{X})\, U .$$

The last relation and proposition 2.2 imply that $f \in U$, and since $U \cap H = (0)$, then $f = 0$.

THEOREM 2.1. *Let* $U \subset M$, $\dim U < \infty$, $\mathscr{F} \in M$. *Then*

$$[D(\mathscr{F}(\mathscr{X})) \cap U^\perp]^\top = U .$$

Proof. Obviously $U \subset [D(\mathscr{F}(\mathscr{X})) \cap U^\perp]^\top$, so it is sufficient to prove the opposite inclusion. Let

$$\omega \in [D(\mathscr{F}(\mathscr{X})) \cap U^\perp]^\top .$$

We set

$$\mathscr{E}_\nu = \{x \in R : |x| < \nu ,\ |\omega(x)| < \nu ,\ |\mathscr{F}(x)| < \nu\} \cap \mathscr{E}'_\nu ,$$

with $\mathscr{E}'_\nu$ as in proposition 2.3. We have

$$\omega \in [H_\nu \cap U^\perp]^\top , \qquad \text{since} \quad H_\nu \cap U^\perp \subset D(\mathscr{F}(\mathscr{X})) \cap U^\perp .$$

Therefore

$$\chi_\nu(\tilde{\mathscr{X}})\, \omega \in [H_\nu \cap U^\perp]^\top , \qquad \text{and since}$$

$$H_\nu \cap U^\perp = H_\nu \ominus \chi_\nu(\tilde{\mathscr{X}})\, U , \qquad \text{we get}$$

$$\chi_\nu(\tilde{\mathscr{X}})\, \omega \in [H_\nu \ominus \chi_\nu(\tilde{\mathscr{X}})\, U]^\top .$$

Because $\chi_\nu(\tilde{\mathscr{X}})\, \omega \in H_\nu$, from the theorem on orthogonal projections we have $\chi_\nu(\mathscr{X})\, \omega \in \chi_\nu(\tilde{\mathscr{X}})\, U$, and by virtue of proposition 2.2, $\omega \in U$.

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