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Simeon Reich

A Remark on Set-Valued Mappings that Satisfy the Leray-Sckauder Condition II

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 13 gennaio 1979

Presiede il Socio Anziano ALESSANDRO ROSSI–FANELLI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Analisi funzionale. — *A Remark on Set-Valued Mappings that Satisfy the Leray-Schauder Condition II.* Nota di SIMEON REICH, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si verificano due congetture dell'Autore.

The purpose of this note, which is a continuation of [4], is to verify Conjectures 3.8 and 4.4 in [3] (in 4.4 (*i*), however, we need to assume that E itself is quasi-complete):

THEOREM. *Let* C *be a closed subset of a locally convex Hausdorff topological vector space* E, *and let an upper semicontinuous* F *assign to each point in* C *a nonempty closed convex subset of* E. *Suppose that* F *has a bounded range, and that there is a point* w *in the interior of* C *such that*

$$(\mathrm{L}-\mathrm{S}) \qquad \text{for every} \quad y \in bdy\,(\mathrm{C}) \quad \text{and} \quad z \in \mathrm{F}(y)\,,$$
$$z - w \neq m\,(y - w) \qquad \text{for all } m > 1\,.$$

If either

(*a*) F (C) *is relatively compact*,

or

(*b*) F *is condensing and* E *is quasi-complete*,

(*) Nella seduta del 13 gennaio 1979.

or

(*c*) F *is condensing with compact point images and* C *is quasi-complete, then* F *has a fixed point.*

The main difference between the present note and [4] is that C is not assumed to be convex. The proof is based on an idea of Hahn [1] and on a fixed point theorem of Himmelberg [2]. It also leads to a partial improvement of [5]. See also [6].

Proof of the Theorem.

We may and shall assume that $w = 0$. Let $K = \{x \in C : x \in tF(x)$ for some $0 \leq t \leq 1\}$. K is nonempty ($0 \in K$) and closed (F is upper semicontinuous). If (*a*) holds, then K is certainly compact. If either (*b*) or (*c*) hold, then since K is a subset of $co(\{0\} \cup F(K))$, it is totally bounded and therefore (C is quasi-complete) also compact. The boundary of C is closed and if F is fixed point free, disjoint from K (By (L — S)). Since E is Hausdorff, it is in fact a gauge space, and therefore there is a continuous $a : E \to [0, 1]$ such that $a(x) = 1$ for $x \in K$ and $a(x) = 0$ for $x \in bdy(C)$. Let G assign to each $x \in E$ the set $a(x) F(x)$ if $x \in C$ and $\{0\}$ if $x \notin C$. G is easily seen to be upper semicontinuous. If (*a*) holds, then its range is clearly relatively compact. If either (*b*) or (*c*) hold, and B is a bounded subset of E, then $G(B) \subset co(\{0\} \cup F(B \cap C))$, so that G is condensing. It also has a bounded range. It follows that there is a convex closed totally bounded subset D of E that is mapped into itself by G. If (*b*) holds, then D is compact. If (*c*) holds, then $C \cap D$ is nonempty and compact, and so are $G(C \cap D)$ and $G(D)$. Thus in all three cases G maps a convex subset of E into a compact subset of itself. Consequently, it has a fixed point z. It belongs to C and $z \in a(z) F(z)$. Thus $z \in K$ and, since $a(z) = 1$, $z \in F(z)$.

Thus the Theorem of [5] remains true for non-convex K if $F(t, \cdot)$ satisfies the Leray-Schauder condition. However, the proofs presented there for the second part of this Theorem (as well as for the Remark) seem to work only if $F(t, \cdot)$ is inward.

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[5] S. REICH - *A random fixed point theorem for set-valued mappings*, «Atti Accad. Naz. Lincei», to appear.

[6] R. SCHÖNEBERG - *Leray-Schauder principles for condensing multi-valued mappings in Fréchet spaces*, to appear.