Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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On the complex differentials $\partial\partial$, $\partial\bar{\partial}$, $\bar{\partial}\bar{\partial}$

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **66** (1979), n.1, p. 19–21.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1979_8_66_1_19_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1979.

Geometria differenziale. — *On the complex differentials* $\partial\partial$, $\partial\bar{\partial}$, $\overline{\partial\partial}$. Nota di GIOVANNI BATTISTA RIZZA, presentata (*) dal Socio E. MARTINELLI.

RIASSUNTO. — L'introduzione dei differenziali ordinari complessi di secondo ordine consente di esprimere in forma compatta e invariante per trasformazioni pseudoconformi, l'angolo che interviene nel contatto tra una ipersuperficie ed una superficie caratteristica di $\mathbf{R}^{2n}$.

1. The present Note [1] introduces the complex ordinary differentials of second order. They appear as Levi–like expressions and result to be invariant under pseudoconformal transformations in $\mathbf{R}^{2n}$. This leads to a compact and invariant formula for the angle, that occurs in the contact between a hypersurface and a holomorphic surface of $\mathbf{R}^{2n}$. Special cases and remarks complete the Note.

The authors of the above mentioned research M. Ferroni and G.B. Rizza intend to dedicate their work to prof. B. Segre. He was the first, who remarked that Levi's invariant expression plays an interesting role in contact problems in $\mathbf{R}^4$.

2. Let $\rho : \mathbf{C}^n \to \mathbf{R}^{2n}$ be the 1 — 1 mapping defined by

$$\rho : (z^1, \cdots, z^n) \mapsto (x^1, \cdots, x^{2n}),$$

where $z^p = x^p + ix^{n+p}$. In the following we make use in $\mathbf{R}^{2n}$ of the isotropic coordinates $z^1, \cdots, z^n, z^{\bar{1}}, \cdots, z^{\bar{n}}$, where $z^{\bar{p}} = \overline{z^p}$. When $\mathbf{R}^{2n}$ is regarded as a vector space on $\mathbf{R}$, we can speak of isotropic components of the vector of $\mathbf{R}^{2n}$.

Let h be a bi-holomorphic mapping of $\mathbf{C}^n$. Then $\rho \circ h \circ \rho^{-1}$ is said a *pseudo-conformal transformation* of $\mathbf{R}^{2n}$.

If $\mathscr{C}$ is a complex curve of $\mathbf{C}^n$, then $\mathrm{S} = \rho(\mathscr{C})$ is a *holomorphic surface* of $\mathbf{R}^{2n}$.

Now, given a twice differentiable real valued function $\varphi(x^1, \cdots, x^{2n}) = \Phi(z^1, \cdots, z^n, z^{\bar{1}}, \cdots, z^{\bar{n}})$, defined in an open set U of $\mathbf{R}^{2n}$, we consider the *complex differentials*

$$\partial\Phi = \Phi_p \, dz^p \,, \; \bar{\partial}\Phi = \Phi_{\bar{p}} \, dz^{\bar{p}} \,, \tag{1}$$

(*) Nella seduta del 13 gennaio 1979.

(1) The topic was object of a short communication hold by G. B. Rizza at the International Congress of Mathematicians (Helsinki, Aug. 1978). An ample paper with proofs and full details will appear in Boll. Un. Mat. Ital., 1979.

where Φ_p, $\Phi_{\bar{p}}$ are the formal derivatives of Φ with respect to the complex variables z^p, $z^{\bar{p}}$. In equations (1) and in the sequel Einstein's summation convention is adopted.

We regard here ∂, $\bar{\partial}$ not as exterior differentials, but as ordinary differentials. Therefore we can consider the *second differentials*

$$\partial\bar{\partial}\,\Phi = \Phi_{p\bar{q}}\,\mathrm{d}z^p\,\mathrm{d}z^{\bar{q}} \tag{2}$$

$$\partial\partial\Phi = \Phi_{pq}\,\mathrm{d}z^p\,\mathrm{d}z^q + \Phi_r\,\mathrm{d}^2 z^r\,;\ \bar{\partial}\bar{\partial}\Phi = \Phi_{\bar{p}\bar{q}}\,\mathrm{d}z^{\bar{p}}\,\mathrm{d}z^{\bar{q}} + \Phi_{\bar{r}}\,\mathrm{d}^2 z^{\bar{r}}\,, \tag{3}$$

where $\Phi_{p\bar{q}} = \dfrac{\partial^2\Phi}{\partial z^p\,\partial z^{\bar{q}}}, \cdots$.

The hybrid differential $\partial\bar{\partial}\Phi$ is nothing else than the well known *Levi–Krzoska* form. Analogous but more complicated expressions are the differentials $\partial\partial\Phi$, $\bar{\partial}\bar{\partial}\Phi$; in particular, if the z^r's are regarded as independent variables, then $\mathrm{d}^2 z^r = \mathrm{d}^2 z^{\bar{r}} = 0$ and the analogy becomes more evident.

Now we can state

THEOREM 1. *The complex differentials* $\partial\Phi$, $\bar{\partial}\Phi$, $\partial\bar{\partial}\Phi$, $\partial\partial\Phi$, $\bar{\partial}\bar{\partial}\Phi$ *are invariant under pseudoconformal transformations in* $\mathbf{R}^{2n}$.

The result is more or less known for $\partial\Phi$, $\bar{\partial}\Phi$, $\partial\bar{\partial}\Phi$, but new for $\partial\partial\Phi$, $\bar{\partial}\bar{\partial}\Phi$.

Let $\mathscr{X}$ be the set of the vector fields of the class C^1 and $\mathscr{F}$ the set of the complex valued functions of the class C^0, defined on U (more generally on a parametric regular subset S (U) of U. Then the complex differentials defined in (1), (2), (3) can be regarded as mappings defined on $\mathscr{X}$ having values in $\mathscr{F}$. For example, for any field v of $\mathscr{X}$ with isotropic components $(v^1,\cdots,v^n,v^{\bar{1}},\cdots,v^{\bar{n}})$ we consider the function $\partial\partial\Phi(v)$ of $\mathscr{F}$, having at point x the value $\partial\partial\Phi(v)_x = \Phi_{pq}\,v^p\,v^q + \Phi_r\,\mathrm{d}v^r(v)$.

3. We come now to some geometrical problems.

Let H be a *hypersurface* of $\mathbf{R}^{2n}$ of the class C^2 through the origin O, regular at point O and locally defined by

$$\varphi(x^1,\cdots,x^{2n}) = \Phi(z^1,\cdots,z^n,z^{\bar{1}},\cdots,z^{\bar{n}}) = 0\,.$$

Let S be a *holomorphic surface* through the origin O, regular at point O and locally defined by

$$z^p = f^p(c) \quad , \quad z^{\bar{p}} = f^{\bar{p}}(c)\,,$$

where c is a complex parameter and $f^{\bar{p}}(c) = \overline{f^p(c)}$.

Assume now that S is *tangent* to H at point O. Then the curve $C = S \cap H$ has O as double point. We can state

THEOREM 2. *The angle* α *between the two tangents to* C *at point* O *is given by*

$$\cos \alpha = \frac{\partial\bar{\partial}\Phi\,(t)_0}{\sqrt{\partial\partial\Phi\,(t)_0\,\bar{\partial}\bar{\partial}\Phi\,(t)_0}}, \tag{4}$$

where t *is the vector field on* S, *having isotropic components* $\mathrm{d}f^p/\mathrm{d}c$ *and their conjugates.*

The geometrical character of the problem assures that the result does not depend on the analytic representations of H and of S.

Note also that the angle α does not change of you substitute a general hermitian metric to the euclidean metric $\mathrm{d}s^2 = \Sigma\,\mathrm{d}z^p\,\mathrm{d}\bar{z}^p$ of $\mathbf{R}^{2n}$.

More interesting is

THEOREM 3. *The angle* α, *that occurs in the contact problem concerning a hypersurface and a holomorphic surface, is invariant under pseudoconformal transformations in* $\mathbf{R}^{2n}$.

The above metioned results lead to consider many special cases and to solve other similar contact problems. In particular, a result of B. Segre concerning the case $n = 2$ can be obtained as special case.

Remark also that a necessary and sufficient condition that the two tangent of C at O coincide is given by $\partial\partial\Phi\,(t)_0\,\overline{\partial\partial}\Phi\,(t)_0 - (\partial\bar{\partial}\Phi\,(t)_0)^2 = 0$. Consider now the expression $\partial\partial\Phi\,(v)_0\,\overline{\partial\partial}\Phi\,(v)_0 - (\partial\bar{\partial}\Phi\,(v)_0)^2$, where v is a vector field of class C^1 defined on S and tangent to S. This expression, invariant under pseudoconformal transformations, when S is a holomorphic plane and $n = 2$, reduces up to a positive factor to an expression introduced by B. Segre in 1932.