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P.N. Pandey

Groups of conformal transformations in conformally related Finsler manifolds

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Geometria differenziale. — *Groups of conformal transformations in conformally related Finsler manifolds* (*). Nota di P. N. PANDEY, presentata (**) dal Socio E. MARTINELLI.

RIASSUNTO. — L'esistenza di varietà Riemanniane in relazione conforme che ammettano moti conformi è stata studiata da M. S. Knebelman [2] (1), K. Yano [4], e G. H. Katzin e Jack Levine [1].

Scopo di questo lavoro è stabilire l'esistenza di varietà di Finsler in relazione conforme che ammetano trasformazioni conformi. Le notazioni di questo lavoro differiscono leggermente da quelle di H. Rund [3].

1. INTRODUCTION

Let F_n be a Finsler manifold of class at least C^6 equipped with line elements $(x^i, \dot{x}^i)$ and the metric function $F(x^i, \dot{x}^i)$ (2) satisfying the required conditions [3]. The metric tensor defined by

$$g_{ij} = \tfrac{1}{2}\dot{\partial}_i \dot{\partial}_j F^2 \text{ (3)}, \tag{1.1}$$

is positively homogeneous of degree zero in $\dot{x}^i$'s and symmetric in its indices. The tensor C_{ijk} defined by

$$C_{ijk} = \tfrac{1}{2}\dot{\partial}_k g_{ij}, \tag{1.2}$$

is symmetric in all its lower indices and satisfies

$$C_{ijk}\dot{x}^i = C_{jik}\dot{x}^i = C_{jki}\dot{x}^i = 0. \tag{1.3}$$

Introducing the connection parameters G^i_{jk}, Berwald defined the covariant derivative $\nabla_k X^i$ of a vector X^i:

$$\nabla_k X^i = \partial_k X^i - (\dot{\partial}_r X^i) G^r_k + X^r G^i_{rk}, \tag{1.4}$$

where

$$G^i_j \overset{\text{def}}{=} G^i_{jk}\dot{x}^k. \tag{1.5}$$

(*) This work was orally presented by the author at the forty-sixth annual session of the National Academy of Sciences, India, held on February 5, 1977, at Delhi University India. Now the author wishes to dedicate this paper to Late Prof. B. Segre.

(**) Nella seduta del 16 dicembre 1978.

(1) Numbers in square brackets refer to the references given at the end of the paper.

(2) Henceforward all the geometric objects are assumed to be functions of $(x^i, \dot{x}^i)$ unless otherwise stated, and the indices $i, j, k, \cdots$ run over positive integers 1 to n.

(3) $\dot{\partial}_i \equiv \partial/\partial\dot{x}^i$, $\partial_i \equiv \partial/\partial x^i$.

Let us consider another Finsler manifold $\bar{F}_n$ with metric function $\bar{F}$ and metric tensor $\bar{g}_{ij}$. The Finsler manifolds F_n and $\bar{F}_n$ are called conformally related if there exists a factor of proportionality μ between the two metric tensors g_{ij} and $\bar{g}_{ij}$:

$$\bar{g}_{ij} = \mu g_{ij} . \tag{1.6}$$

Knebelman proved that the factor of proportionality μ is at most a point function and we may now write (1.6) in the form

$$\bar{g}_{ij} = e^{2\sigma} g_{ij} , \tag{1.7}$$

where

$$\sigma = \sigma(x) = \tfrac{1}{2} \log \mu , \tag{1.8}$$

so that

$$a)\ \bar{g}^{ij} = \bar{e}^{2\sigma} g^{ij} , \qquad b)\ \bar{F}^2 = e^{2\sigma} F^2 . \tag{1.9}$$

It can be easily verified that the Berwald connection parameters $\bar{G}^i_{jk}$ and G^i_{jk} for the above two Finsler manifolds are connected by

$$\bar{G}^i_{jk} = G^i_{jk} - B^{im}_{jk} \sigma_m , \tag{1.10}$$

where

$$B^{im} = \tfrac{1}{2} F^2 g^{im} - \dot{x}^i \dot{x}^m \quad , \quad B^{im}_{jk} = \dot{\partial}_j \dot{\partial}_k B^{im} \quad , \quad \sigma_m = \partial_m \sigma \tag{1.11}$$

and

$$B^{im}_j = \tfrac{1}{2} (\dot{\partial}_j F^2) g^{im} + \tfrac{1}{2} F^2 C^{im}_j - \delta^i_j \dot{x}^m - \delta^m_j \dot{x}^i . \tag{1.12}$$

An infinitesimal transformation

$$\bar{x}^i = x^i + \epsilon v^i(x) \tag{1.13}$$

is called a conformal transformation if the Lie derivative of the metric tensor g_{ij} is proportional to itself i.e. if there exists a scalar point function ρ satisfying

$$\mathcal{L} g_{ij} = \rho g_{ij} . \tag{1.14}$$

The above transformation is called homothetic if ρ is a constant. The above transformation is called motion if and only if $\rho = 0$. Thus for a homothetic transformation

$$\mathcal{L} g_{ij} = c g_{ij} , \tag{1.15}$$

where c is some constant, and for a motion we have

$$\mathcal{L} g_{ij} = 0 . \tag{1.16}$$

Thus we see that homothetic transformation and motion are particular cases of a general conformal transformation. If ρ is neither constant nor zero then the conformal transformation, is called a proper conformal transformation.

2. GROUPS OF CONFORMAL TRANSFORMATIONS

Let $\overline{F}_n$ admits a group G_r of conformal transformations generated by r vectors $\bar{v}^i_\alpha$, $(\alpha = 1, 2, \cdots, r)$ such that

$$\mathscr{L}_\alpha \bar{g}_{ij} = \overline{\psi}_\alpha \bar{g}_{ij} . \tag{2.1}$$

Transvecting (2.1) with $\dot{x}^j$ we have

$$\mathscr{L}_\alpha \dot{\partial}_i \overline{F}^2 = \overline{\psi}_\alpha \dot{\partial}_i F^2. \tag{2.2}$$

Differentiating (2.2) partially with respect to $\dot{x}^j$ and using (2.1) we have

$$\dot{\partial}_j \overline{\psi}_\alpha = 0 , \tag{2.3}$$

i.e. $\overline{\psi}_\alpha$ are functions of the position coordinates only. I want to discuss the question whether F_n admits a group of general conformal transformations, and I propose the following

THEOREM 2.1. *If F_n and $\overline{F}_n$ are two conformally related Finsler manifolds such that $\overline{F}_n$ admits an r-parameter group $\overline{G}_r$ of general conformal transformations (proper conformal transformations, homothetic transformations or motions), then F_n admits the same group $\overline{G}_r$ as a group of (in general) proper conformal transformations.*

Proof. Let $\overline{F}_n$ admit an r-parameter group $\overline{G}_r$ of general conformal transformations generated by $\bar{v}^i_\alpha$, $(\alpha = 1, 2, \cdots, r)$ satisfying (2.1). Transvection of (2.1) with $\dot{x}^i \dot{x}^j$ yields

$$\mathscr{L}_\alpha \overline{F}^2 = \overline{\psi}_\alpha \overline{F}^2 . \tag{2.4}$$

In (2.4), using

$$\mathscr{L}_\alpha P = \bar{v}^m_\alpha \overline{\nabla}_m P + (\dot{\partial}_m P) \overline{\nabla}_s \bar{v}^m_\alpha \dot{x}^s , \tag{2.5}$$

and the fact that the metric function is a covariant constant in the sense of Berwald we have

$$(\dot{\partial}_m \overline{F}^2) \overline{\nabla}_s \bar{v}^m_\alpha \dot{x}^s = \overline{\psi}_\alpha \overline{F}^2 . \tag{2.6}$$

The covariant derivative $\overline{\nabla}_s \bar{v}^m$ of a vector $\bar{v}^m$ is given by

$$\overline{\nabla}_s \bar{v}^m = \partial_s \bar{v}^m + \bar{v}^1 \overline{G}^m_{1s} , \tag{2.7}$$

here we have used the fact that $\bar{v}^1$ is independent of $\dot{x}^i$'s. Using (1.10) in (2.7) we have

$$\overline{\nabla}_s \bar{v}^m = \nabla_s \bar{v}^m - \bar{v}^1 B^{mp}_{1s} \sigma_p . \tag{2.8}$$

In view of (2.8) we may write

$$\overline{\nabla}_s \bar{v}_\alpha^m = \nabla_s \bar{v}_\alpha^m - \bar{v}_\alpha^1 \, B_{1s}^{mp} \, \sigma_p \, . \tag{2.9}$$

Using (2.9) in (2.6) we have

$$(\dot{\partial}_m \overline{F}^2) \{\nabla_s \bar{v}_\alpha^m - \bar{v}_\alpha^1 \, B_{1s}^{mp} \, \sigma_p\} \, \dot{x}^s = \overline{\psi}_\alpha \overline{F}^2 \, . \tag{2.10}$$

Using (1.9 *b*) in (2.10) we have

$$(\dot{\partial}_m F^2) \{\dot{x}^s \, \nabla_s \bar{v}_\alpha^m - \bar{v}_\alpha^1 \, B_1^{mp} \, \sigma_p\} = \overline{\psi}_\alpha \, F^2 \, . \tag{2.11}$$

From (1.12) and (2.11) we can deduce

$$(\dot{\partial}_m F^2) \, \nabla_s \bar{v}_\alpha^m \, \dot{x}^s = (\overline{\psi}_\alpha - 2 \, \bar{v}_\alpha^1 \, \sigma_1) \, F^2 \, ,$$

which implies

$$\mathscr{L}_\alpha \, F^2 = \psi_\alpha \, F^2 \tag{2.12a}$$

where

$$\psi_\alpha = \overline{\psi}_\alpha - 2 \, \bar{v}_\alpha^1 \, \sigma_1 \, . \tag{2.12b}$$

Differentiating (2.12a) twice partially with respect to $\dot{x}$'s, using (1.1), and the commutation formula exhibited by

$$\dot{\partial}_1 \mathscr{L} T_j^i - \mathscr{L} \dot{\partial}_1 \, T_j^i = 0 \, ,$$

we have

$$\mathscr{L}_\alpha g_{ij} = \psi_\alpha \, g_{ij} \, ,$$

which proves the statement.

3. Cases when G_r in F_n is a group of homothetic transformations

In this article we shall discuss the possibility that F_n admits G_r as group of homothetic transformations. There arises three cases:

(*a*) G_r admitted by $\overline{F}_n$ is a group of motions,

(*b*) G_r admitted by $\overline{F}_n$ is a group of homothetic transformations,

(*c*) G_r admitted by $\overline{F}_n$ is a group of proper conformal transformations.

We shall discuss the above cases one by one.

Case (a). In this case we assume that $\overline{F}_n$ admits a group G_r of motions and want to know whether there exists a F_n which admits G_r as a group of homothetic transformations, and propose the

THEOREM 3.1. *If a $\overline{F}_n$ admits an r-parameter group G_r of motions such that the rank of the generator-matrix $[\bar{v}_\alpha^i]$ is $r < n$, and the rank of the structure-constant matrix $[C_{\alpha\beta}^\gamma]$, (γ = column, α, β = row] is $< r$, then there exists a Finsler manifold F_n conformal to $\overline{F}_n$ admitting G_r as a group of homothetic transformations.*

Proof. Let $\overline{F}_n$ admits an r-parameter group G_r of motions such that

$$(3.1) \qquad \mathcal{L}_\alpha \bar{g}_{ij} = 0 ,$$

i.e. $\overline{\psi}_\alpha = 0$ in (2.1). If F_n admits G_r as group of homothetic transformations, then

$$(3.2) \qquad \psi_\alpha = K_\alpha = \text{constant} ,$$

such that at least one $K_\alpha \neq 0$. Then from (2.12b) we have $\psi_\alpha = K_\alpha = -2\,\bar{v}^1_\alpha\,\sigma_1$, which implies

$$(3.3) \qquad \mathcal{L}_\alpha \sigma = -\tfrac{1}{2} K_\alpha .$$

In view of the commutation formula

$$(3.4) \qquad (\mathcal{L}_\alpha \mathcal{L}_\beta - \mathcal{L}_\beta \mathcal{L}_\alpha)\,\sigma = C^\gamma_{\alpha\beta} \mathcal{L}_\gamma \sigma ,$$

and the fact that Lie derivative of any constant is zero, (3.3) gives

$$(3.5) \qquad C^\gamma_{\alpha\beta} K_\gamma = 0 .$$

Equation (3.3) will be integrable if (3.5) have non-trivial solutions for K_γ, which is possible if the rank of the matrix $[C^\gamma_{\alpha\beta}] < r$.

Case (b). Let us assume that $\overline{F}_n$ admits a group G_r of homothetic transformations. We want to discuss whether there exists F_n admitting G_r as group of homothetic transformations and we have

THEOREM 3.2. *If $\overline{F}_n$ admits a group G_r of homothetic transformations such that the rank of $[\bar{v}^i_\alpha]$ is $r < n$, then there exists a Finsler manifold F_n conformal to $\overline{F}_n$ admitting G_r as group of homothetic transformations.*

Proof. From the hypothesis of this theorem we have

$$\overline{\psi}_\alpha = \overline{C}_\alpha = \text{constant (at least one } \overline{C}_\alpha \neq 0) ,$$

and

$$\psi_\alpha = C_\alpha = \text{constant (at least one } C_\alpha \neq 0) .$$

Utilizing the commutation formula (3.4), we can see that in this case we have

$$(3.6) \qquad C^\gamma_{\alpha\beta} \overline{C}_\gamma = 0 ,$$

so the rank of $[C^\gamma_{\alpha\beta}] < r$. From (2.12b) we have

$$2\,\bar{v}^1_\alpha\,\sigma_1 = \overline{C}_\alpha - C_\alpha$$

which can be written as

$$(3.7) \qquad 2\mathcal{L}_\alpha \sigma = \overline{C}_\alpha - C_\alpha .$$

Taking the Lie derivative with respect to $\bar{v}^i_\beta$ and using (3.4) and (3.6) we have $C^\gamma_{\alpha\beta} C_\gamma = 0$, which is an integrability condition for (3.7). Since the rank of $[\bar{v}^1_\alpha]$ is $r < n$ and rank of $[C^\gamma_{\alpha\beta}] < r$ (as we have proved), (3.7) will always admit a non-trivial solution for the C_γ (and such that $\bar{C}_\gamma \neq C_\gamma$ for at least one γ). Thus we have the theorem.

Case (*c*) In this case $\bar{F}_n$ admits a group G_r of proper conformal transformations and want to find F_n admitting G_r as a group of homothetic transformations. In this case we have

THEOREM 3.3. *If a* $\bar{F}_n$ *admits an r-parameter group* G_r *of proper conformal transformations such that the rank of* $[\bar{v}^1_\alpha]$ *is* $r < n$, *and the rank of matrix* $[C^\gamma_{\alpha\beta}]$ *is* $< r$, *then there will exist manifolds* F_n *conformal to* $\bar{F}_n$ *for which* G_r *will be a group of homothetic transformations.*

Proof. Under the hypothesis of this theorem we have

$$\mathscr{L}_\alpha \bar{g}_{ij} = \bar{\psi}_\alpha \bar{g}_{ij} \qquad \text{(at least one } \bar{\psi}_\alpha \neq 0) . \tag{3.8}$$

From (2.12b) we have

$$2\mathscr{L}_\alpha \sigma = \bar{\psi}_\alpha - C_\alpha . \tag{3.9}$$

From (3.8) we have

$$\mathscr{L}_\alpha \bar{\psi}_\beta - \mathscr{L}_\beta \bar{\psi}_\alpha = C^\gamma_{\alpha\beta} \bar{\psi}_\gamma . \tag{3.10}$$

Using (3.10) in the integrability condition of (3.9) we have $C^\gamma_{\alpha\beta} C_\gamma = 0$, which will admit a non trivial solution for C_γ if $[C^\gamma_{\alpha\beta}] < r$. Thus we have the theorem.

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